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Combinatorial maps

Tutorial

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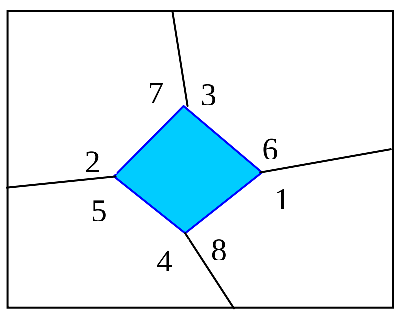

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Combinatorial maps

Permutations

Let us denote by C finite set of elements. Let us call elements of this set either simply elements or points or corners. Let us denote elements with small Greek letters. The cardinality of C is n. Let us call permutation on C one-one map from C to C. In permutation P let us denote target of α by α^P. Thus, we may use for permutation P following denotations:

$$P = \begin{pmatrix} 1 & 2 & \dots & n \\ 1^P & 2^P & \dots & n^P \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha^P \end{pmatrix}.$$

We define multiplication of two permutations P and Q denoting it $P \cdot Q$ as permutation on C by formula $\alpha^{PQ} = \left(\alpha^P\right)^Q$. Thus we are multiplicating (or are reading multiplication) from left to right. Since multiplication of two permutations is always a permutation then all permutations on C shape a group with respect to multiplication operation that is called permutation group and denoted by S^C (or S_n). As is well known the permutation group is full (in sense faithful) representation of symmetry group in general. Then S^C designates symmetry (or permutation) group's action on C and S_n the symmetry group itself. Permutation group's identity element is identity permutation, that is denoted by I or id, which has identity map on C, i.e., it leaves all elements of C on place. By P^{-1} we designate reverse permutation of P: if P maps α to β, then reverse permutation P^{-1} maps β to α.

Simplest forms of permutation's coding are two row matrix coding or permutation in cyclical coding form, e.g.:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 8 & 5 & 3 & 1 & 6 & 2 \end{pmatrix} = (1\,7\,6)(2\,4\,5\,3\,8).$$

Thus, matrix form is simply (α, α^P) depicted as two row matrix, but cyclical form runs elements according their cyclical order. How to attain these forms of codings of permutation? It is shown in the table:

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 8 & 5 & 3 & 1 & 6 & 2 \end{pmatrix}$$

$$1^P = 7$$
$$7^P = 6$$
$$6^P = 1$$
$$2^P = 4$$
$$4^P = 5$$
$$5^P = 3$$
$$3^P = 8$$
$$8^P = 2$$

$$P = (1\,7\,6)(2\,4\,5\,3\,8)$$

$$(176) = (1, 1^P = 7, 7^P = 6, 6^P = 1)$$

$$(24538) = (2, 2^P = 4, 4^P = 5, 5^P = 3, 3^P = 8, 8^P = 2)$$

Indeed, starting with 1, first we find $1^P = 7$, then after $7^P = 6$, and with $6^P = 1$ cycle is closed. Then we take not yet searched element, say, 2 and find that $2^P = 4$ and so on, until all elements are searched in their cyclical order in the permutation. Why we get cycles without repetitions and not something else? This is because the fact that permutation is bijection (one-one map). Cyclical form (of coding) of permutation is more convenient in general use, because we may give only these elements which are moved, but elements that remain on place may not be specified in the code. For example, permutations $(1)(2\,7\,4)(4)(5)(6)(8)$ and (274) are identical, in particular if we do not bother about to indicate that first permutation acts on set of 8 elements. But permutation $(1\ 27364\ \ 173)(4\ \ 11)$ in matrix form we could not have sufficient space on leaf of paper to depict. [The problem in general is faced in computer programs where both ways of coding have their advantages and disadvantages and are to be used alternately where, of course, the cost of operations on both forms is the determinant of the way of coding. On computer, transform from one code to other is linear operation. Similarly, multiplication in matrix form is linear operation and many other operations are linear or near to that. Thus, permutational calculus, and combinatorial maps' calculus can be performed in very powerful fashion.]

It is easy to see that cyclical form of permutation structurally is set of cyclical lists. This means that by changing cycles in the form or cyclically changing elements in a cycle does not change the permutation in general. For example, the permutation in the previous example can be put down in the way $(5\,3\,8\,2\,4)(7\,6\,1)$ too.

Transposition is a permutation which changes two elements in their places but other remains as they are: transposition t has always cyclical form $(\alpha\,\beta)$. Each permutation can be written in the form of multiplication of transpositions $p = (\alpha_1\beta_2)(\alpha_1\beta_2)...(\alpha_n\,\beta_n)$. For example the previous permutation can be given as multiplication of transpositions in the form $(16)(23)(45)(67)(28)(43)$. Performing all multiplications, multiplying from left to right, we should return to the previous (canonical) cyclical form. Try it! It is easy to see that this way of presenting of permutation in the form of multiplication of transpositions is not unique. However, either n is even or odd is invariant of the permutation.

Combinatorial maps

Let us consider the simplest way of defining of the combinatorial map when it has graph on surface in the correspondence.

Let us have set C with $2m$ elements that are called corners. On set C permutations should act, and they are the objects we are going to deal with.

Definition 1.Two permutations P and Q of order n are *distinct*, if $i^P \neq i^Q$ for each $i \in [1...n]$.
Permutation is said to be *involution* if all its cycles are of order two.

Definition 2. Oriented pair (P,Q) of two distinct permutations is called combinatorial map if the multiplication $P^{-1} \cdot Q$ is involution.

Combinatorial map defined in this way is called geometrical combinatorial map.

Let pair (P,Q) be combinatorial map. Multiplication $P^{-1} \cdot Q$ that is involution is called edge rotation and is denoted by π . Thus, π consists of n distinct transpositions.

Definition 3. Let pair (P,Q) be combinatorial map. P is called vertex rotation and Q is called face rotation.

Example 1. Let $P = (123)(45)(678)$ and $Q = (164)(28)(357)$. (P,Q) be combinatorial map with edge rotation π equal to $(15)(26)(38)(47)$.

Exercise 1. Try to connect combinatorial map (P,Q) from exercise 1 with the picture of graph in fig. 1. Hint. Numbers are to be considered as labels of 'corners' and cyclical orders of 'corners' around vertices and faces are to be considered as permutations. Find vertex, face and edge rotations in the picture of the corresponding graph.

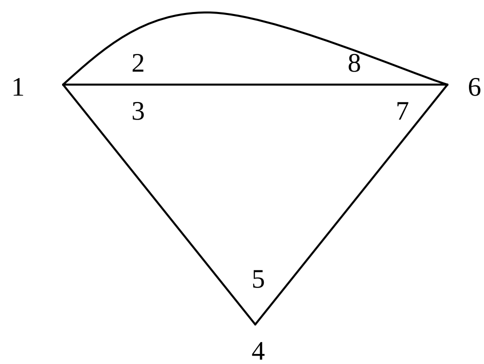


Figure 1.

The correspondence between combinatorial maps and graphs on surfaces.

Let graph G be embedded on oriented surface S. This means that for each vertex $v \in V(G)$ the edges that are adjacent with this vertex are cyclically ordered around it, and in the same time the cyclical order of incident faces on S around vertex v is given too.
Let $N(v)$ be set of neighbors of vertex v. Cyclical order of edges around v induces cyclical order in the set $N(v)$.

Let us denote the cyclically ordered sequence of neighbors of v by $Adj(v) = (v_1, v_2, ..., v_n)$, where n is number of neighbors. Induced cyclical sequence $((v,v_1),(v,v_2),...,(v,v_n))$, that consists from oriented edges around vertex v in the same way characterizes the embedding of the neighborhood of v on the surface S. Outgoing (oriented) edges of each vertex form such cyclical sequence and outgoing (oriented) edges of all vertices form a permutation P (because each such oriented edge is unique). That means that the embedding of the graph on surface S is uniquely determined by the permutation $(=P)$ that acts on set of oriented edges E_G. Let us denote this permutation by P_G^S $(=P)$. Thereby, the embedding of G on S can be given or fixed as $(G,S)=(V,E,\ P_G^S)$.

Let us form one more permutation that characterizes the graph's G embedding on the surface S. Let us consider arbitrary face f with border (as sequence of vertices) $(v_1,v_2,...,v_l)$, where the face f is passed round in the direction opposite to clockwise. Thus, f is oriented anticlockwise and can be characterizes by cyclical oriented sequence of oriented edges i.e., $f = ((v_1, v_2), (v_2, v_3),..., (v_l, v_1))$. Cyclical oriented sequences of all faces form induce a permutation that acts on set of oriented edges E_G. Indeed, if arbitrary oriented edge goes into border of the face, then only once and only in the border of this face. By the same reason each oriented edges goes into border at least one face. Let us denote this permutation with Q_G^S. Thus, embedding of G on S can be given as $(G,S)=(V,E,\ Q_G^S)$. This is equivalently with the previous way of determining the embedding of G on S. Thus, graph G may be fixed on S either by permutation P_G^S or Q_G^S. More radical discovery tells us that graph on surface may be determined without specifying sets V and E at all, but specifying only pair of permutations $(P,\ Q)$ that is isomorphic to $(P_G^S,\ Q_G^S)$. Let us call this mathematical fact Hefter-Edmonds theorem. It was contrived by Hefter as early as 1898, but in contemporary form proved by Edmonds in 1960.

Combinatorial maps. Systematic insight.

Let P,Q be two permutations. Combinatorial map (P,Q) is called geometrical if $P\cdot Q^{-1}$ is involution without fixed points.

Proposition 1. Geometrical map has even number of corners.

Proof. If map has odd number of edges then J $P \cdot Q^{-1}$ is involution with odd number of elements that has at least one fixed element.

Let us call involution $P \cdot Q^{-1}(= \pi)$ *inner edge rotation* but $Q \cdot P^{-1}(= \rho)$ – *edge rotation*. We shall see further the difference between both clearer.

Pair of corners (s,t) is called edge if $s^{P \cdot Q^{-1}} = t$. Let us denote this edge by st.

Proposition 2. If st is edge and belongs to ρ, then st^P belongs to π.

Let us prove that $\rho^P = \pi$. From that should follow what is asserted. Indeed:

$$\rho^P = P^{-1} \cdot \rho \cdot P = P^{-1} \cdot Q \cdot P^{-1} \cdot P = P^{-1} \cdot Q = \pi .$$

Let us call st^P *inner edge*, st being simply edge. If ρ is edge rotation for a map, then π is inner edge rotation. We have proved that application of vertex rotation to edge rotation gives inner edge rotation. Symmetric expression holds too: $\rho^Q = \pi$. Indeed:

$$\rho^Q = Q^{-1} \cdot \rho \cdot Q = Q^{-1} \cdot Q \cdot P^{-1} \cdot Q = P^{-1} \cdot Q = \pi$$

It is easy to see that following expressions holds too: $\rho \cdot P = P \cdot \pi$ and $\rho \cdot Q = Q \cdot \pi$.

It is convenient to observe some terminology. Thus we are speaking that P, Q, π are acting on set C of corners or elements. This set C of $2m$ elements is divided by π into m pairs. Let (c_1, c_2) be such pair that $c_1^{\pi} = c_2$. Provided (c_1, c_2) already belongs to inner edge rotation then there exists a pair (c, c') that belongs to edge rotation, that holds $(c, c') \cdot P = P \cdot (c_1, c_2)$. It may be very convenient not to consider combinatorial maps separately with variable inner edge rotations but as classes of maps with one fixed inner edge rotation π. For such fixed π map $(P,Q) = (P, P \cdot \pi)$ has its unique edge rotation ρ, that holds $\rho \cdot P = P \cdot \pi$. Thus, π is common for a class, but ρ is depending from a map $(P,Q)(= (P, P \cdot \pi))$. In its turn, it becomes quite clear that map (P,Q) is determined only by one rotation, namely, vertex rotation P.

We should keep in mind:

$$\pi = P^{-1} \cdot Q \qquad = Q^{-1} \cdot P$$

$$\rho = Q \cdot P^{-1} \qquad = P \cdot Q^{-1}$$

Map's mirror reflection and dual map.

What looks like combinatorial map's (P, Q) mirror reflection?

Proposition 1. Map (P, Q) has symmetric reflection's map (P^{-1}, Q^{-1}) in correspondence.

Indeed. Mirror reflected vertex and face rotations change their directions from, say, clockwise to anticlockwise [for vertex rotation] and vice versa [for face rotation]. Thus, we could write $P_{mir-sym} = P^{-1}$ and $Q_{mir-sym} = Q^{-1}$.

Proposition 2. Maps π and ρ change their places in mirror reflection in the way $\pi_{mir-sym} = \rho$ and $\rho_{sp-sym} = \pi$.

Indeed, from $(P, Q)_{mir-sym} = (P^{-1}, Q^{-1})$ follow that $\pi_{mir-sym} = P \cdot Q^{-1} = \rho$ and $\rho_{sp-sym} = P^{-1} \cdot Q = \pi$. [We used the fact that multiplication's reflected map is reverse map, i.e., $(A.B)_{mir-sym} = (A.B)^{-1} = B^{-1}.A^{-1}$]. Thus, in mirror reflection rotations π and ρ change as if in their places.

Trying to define map's (P, Q) dual map simply as (Q, P) we use its geometrical interpretation. Thus we write $(P, Q)_{dual} = (Q, P)$. What we get for edge rotations? If in rotation π_{dual} we change P and Q in their places, then rotation itself does not change, thus, $\pi_{dual} = \pi$. $\rho_{dual} = \rho$. But maps themselves, namely, dual map and reflected map, they are different.

Let C be universal set of corners and π be fixed. We are in the class of maps, say, K. Map (P, Q) may be designated with one letter, namely, P. Thus, we have P and would like to have $P_{mir-sym}$ and P_{dual} too. But we may take in our class of maps only those which have the same rotation π. Thus $P_{mir-sym}$ is not member of K and we are going to define what could be called reverse map, that is now within K. Thus reverse map P^{-1} or P_{rev} is defined as $(P^{-1}, P^{-1} \cdot \pi)$. It has, of course, $\pi_{rev} = \pi$ and

$$\rho_{rev} = P^{-1} \cdot (P^{-1}Q^{-1}P)^{-1} = P^{-1} \cdot P^{-1} \cdot Q \cdot P \;\; = P^{-1} \cdot Q^{P} .$$

Correspondence between permutations' and combinatorial maps' classes

Let class K_π have all combinatorial maps with inner edge rotation π fixed. Then class's members may be characterized with pairs (P,π) or (Q,π) or, keeping π in mind, only with one permutation, say, that of vertex rotation, P, where Q may be always calculated using formula $Q = P \cdot \pi$.

Thus, living within fixed rotation π or saying that we live within K_π, every permutation has its combinatorial map in correspondence. Identical permutation e has map (e,π) in correspondence that in its turn has graph with m isolated edges. Transposition $tr = (a,b)$ has map $(tr,\pi) = ((a,b),\pi)$ and star graph S_2 in correspondence.

Multiplication of combinatorial maps.

Let us define multiplication of maps in the class K_π. Let us permutation multiplication take as the base operation. Let us try to define multiplication of maps in the way that class K_π is closed against this operation. Let (P_1,Q_1) and (P_2,Q_2) to class of maps (P,Q) with fixed inner edge rotation π, i.e., to class K_π .

Definition. We define multiplication of two maps in the way:

$$(P_1,Q_1) \times (P_2,Q_2) = (P_3,Q_3)$$

where $P_3 = P_1 \cdot P_2$ and $Q_3 = P_1 \cdot Q_2$

It is easy to see that map (P_3,Q_3) belongs to class K_π: $P_3^{-1} \cdot Q_3 = P_2^{-1} \cdot P_1^{-1} \cdot P_1 \cdot Q_2 = \pi$.

We may write

$$(P_1,Q_1) \times (P_2,Q_2) = (P_1 \cdot P_2,\; P_1 \cdot Q_2) .$$

We may write in more symmetric way too:

$$(P_1,Q_1) \times (P_2,Q_2) = (P_1 \cdot P_2,\; Q_1 \cdot \pi \cdot Q_2) .$$

Further, multiplying maps, we may treat maps as permutations remembering that behind permutations we have maps.

Actually, we may be free in the interchange between maps and permutations within K_π, because of the one-one correspondence between them. We even have more. Every theorem that holds for permutations has its meaning in maps' interpretation too. Thus every permutational theorem has combinatorial map's theorem in correspondence.

Further we are not going to use different designators for multiplication of maps and permutations.

Permutations constitute symmetric group S_{2m}, which acts on corner set C. Group S_{2m} has group $K_m (= K_\pi)$ in correspondence .

Normalized combinatorial maps

Practically working with combinatorial maps with fixed inner edge rotation π it would be convenient to choose some fixed π 'for all cases of life'. We have chosen π equal with $(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)\cdots(2k-1\ \ 2k)$, id est, $k-th$ inner edge is equal to $(2k-1\ \ k)$. Let us further accept this form of π and use in all cases with excuse in cases when it is necessary.

If P is given it is easy to find Q: if in P element $a \in C$ has $b = a^P$ in correspondence, then for Q corresponding element is equal to $a-1$, if a is even, and equal to $a+1$, if a is odd. For example, for given $P = (1\ 7\ 5\ 3\ 6\ 2\ 4\ 8)$ let us calculate Q: $1^P = 7$, gives $1^Q = 8$; cyclically continuing:, $8^P = 1$, gives $8^Q = 2$; further $2^P = 4$, gives $2^Q = 3$; $3^P = 6$, gives $3^Q = 5$; $5^P = 3$, gives $5^Q = 4$; $4^P = 8$, gives; $4^Q = 7$, $7^P = 5$, gives $7^Q = 6$; at last, $6^P = 2$, gives $6^Q = 1$ and cycle is closed. We got: $Q = (1\ 8\ 2\ 3\ 5\ 4\ 7\ 6)$. Let us show these operations in the table:

$$P = (1\ 7\ 5\ 3\ 6\ 2\ 4\ 8)$$
$$1^P = 7 \Rightarrow 1^Q = 8$$
$$8^P = 1 \Rightarrow 8^Q = 2$$
$$2^P = 4 \Rightarrow 2^Q = 3$$
$$3^P = 6 \Rightarrow 3^Q = 5$$
$$5^P = 3 \Rightarrow 5^Q = 4$$
$$4^P = 8 \Rightarrow 4^Q = 7$$
$$7^P = 5 \Rightarrow 7^Q = 6$$
$$6^P = 2 \Rightarrow 6^Q = 1$$
$$Q = (1\ 8\ 2\ 3\ 5\ 4\ 7\ 6)$$

It is easy to see in 3-permutation where second row shows P and third row shows Q:

$$\begin{matrix}(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)\\(7\ 4\ 6\ 8\ 3\ 2\ 5\ 1)\\(8\ 3\ 5\ 7\ 4\ 1\ 6\ 2)\end{matrix}.$$

Exercise 1. Find Q for given $P = (1\ 8\ 2\ 3\ 5\ 4\ 7\ 6)$ using the method given above.

Exercise 2. Find Q, for given $P = (1\ 8\ 6\ 3)(2\ 7\ 5\ 4)$.

Hint: if cycle ends start new cycle taking not yet searched element.

Let us calculate edge rotation of normalized map using formula $\rho = \pi^{P^{-1}}$:

$$P = (1\ 7\ 5\ 3\ 6\ 2\ 4\ 8)$$
$$(1\ 2)^{P^{-1}} = (8\ 6)$$
$$(3\ 4)^{P^{-1}} = (5\ 2)$$
$$(5\ 6)^{P^{-1}} = (7\ 3)$$
$$(7\ 8)^{P^{-1}} = (1\ 4)$$
$$\rho = (1\ 4)(2\ 5)(3\ 7)(6\ 8)$$

Let us calculate edge rotation of normalized map, namely, using formula $P \cdot Q^{-1}$ and previous method of calculating Q in the way [prove that it is correctly]:

$$
\begin{gathered}
P = (1\ 7\ 5\ 3\ 6\ 2\ 4\ 8) \\
1^P = 7 \;\; 8^{P^{-1}} = 4 \;\Rightarrow\; (1\ 4) \in \rho \\
2^P = 4 \;\; 3^{P^{-1}} = 5 \;\Rightarrow\; (2\ 5) \in \rho \\
3^P = 6 \;\; 5^{P^{-1}} = 7 \;\Rightarrow\; (3\ 7) \in \rho \\
6^P = 2 \;\; 1^{P^{-1}} = 8 \;\Rightarrow\; (6\ 8) \in \rho \\
\rho = (1\ 4)(2\ 5)(3\ 7)(6\ 8)
\end{gathered}
$$

Exercise 3. Calculate edge rotation ρ for maps $P_1 = (1\ 8\ 2\ 3\ 5\ 4\ 7\ 6)$ and $P_2 = (1\ 8\ 6\ 3)(2\ 7\ 5\ 4)$, using the method given above!

Geometrical interpretation of combinatorial maps.

Combinatorial map may be interpreted as graph on orientable surface. The fact corresponds to Hefter-Edmonds theorem.

In place to try to prove this theorem we suggest following construction which shows what is behind this theorem and how combinatorial maps may be considered as graphs on surfaces.

Constructive assumption. Using one general method, to arbitrary combinatorial map a graph may be mapped that is drawn in the plane with edge crossings in general.

Note. Any graph may be drawn in the plane with edge crossing in general.

Construction. Let P be vertex rotation of the combinatorial map. Each cycle of length l let us picture as a vertex with l halfedges, between which we put numbers of corners clockwise in the order corresponding to this cycle. Arbitrary inner edge $(a\ \ b)$ of the map $(P, P\cdot\pi)$ we picture in the plane followingly: let us find halfedge ε_a that is before corner with number a moving clockwise; likewise let us find ε_b; let us connect halfedges ε_a and ε_b with a non crossing curve, but allowing crossings with other similar connections if necessary. Corresponding edge to this inner edge should be $(a^{P^{-1}} b^{P^{-1}})$, where $a^{P^{-1}}$ follows anticlockwise a and $b^{P^{-1}}$ follows anticlockwise b in P. Iteratively applying this operation, we get map pictured in the plane in the sense of this construction.

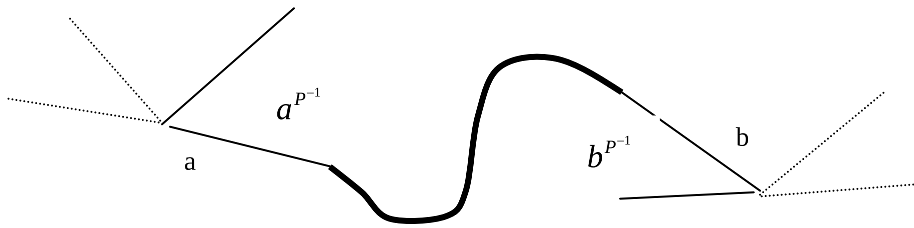


Illustration of connection of inner halfedges $(a\ \ b)$ and halfedges $(a^{P^{-1}} b^{P^{-1}})$ in the drawing of the map in plane in case of different vertices.

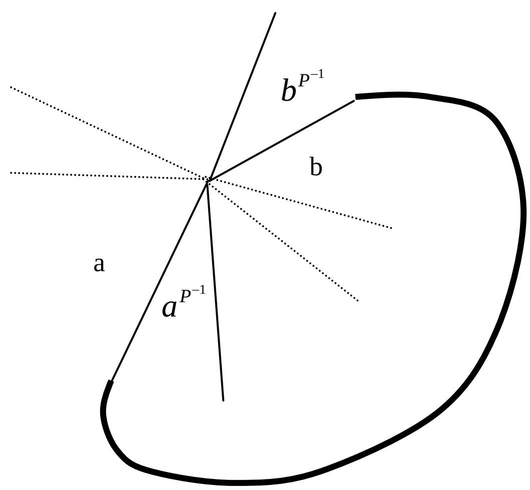
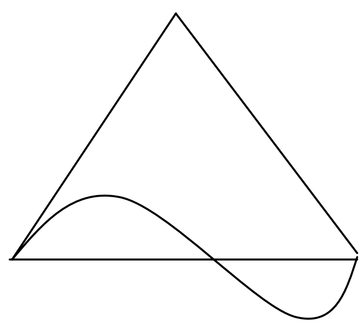


Illustration of connection of inner halfedges $(a\ \ b)$ end halfedges $(a^{P^{-1}} b^{P^{-1}})$ in the drawing of the map in plane in case of the same vertex.

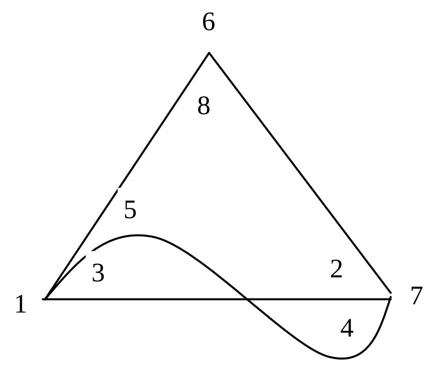
Drawing of the graph corresponding to combinatorial map

From the previous chapter we know that combinatorial maps have nice interpretation as graph's embedding in the plane with edge crossing and Hefter-Edmonts theorem in its turn says that these graph's embeddings with crossings in the plane have graph's embeddings without crossings on oriented surfaces with sufficiently large genus. We gave a construction how to picture map in the plane and this same construction may be used as a practical tool whenever we want to such drawing.

In order to get clear percept what we are doing, let us behave conversely and let us find for a given graph's embedding corresponding eventual combinatorial map. Below we have graph embedded in the plane [with two edges crossing]. We would say that this graph is given with fixed rotation [of edges, as is told traditionally, or of faces, following theory above].



Let us add labels to this drawing to corners between edges.



After doing this a combinatorial map is already fixed, and let us write it down. First let us write vertex rotation $(1\ 5\ 3)(2\ 7\ 4)(6\ 8)$, running corners around vertices clockwise. Further its face rotation should be $(1\ 6\ 7\ 3\ 2\ 8\ 5\ 4)$. Yes, we get only one face because of crossing of two edges in the plane. Further edge rotation should be $(1\ 8)(2\ 6)(3\ 4)(5\ 7)$. Inner edge rotation is $(1\ 2)(3\ 4)(5\ 6)(7\ 8)$. What we did to get just this inner edge rotation in order to get normalized combinatorial map? Find it out.

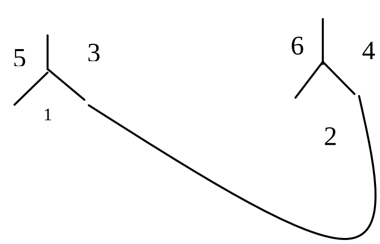
Take note how map's edge rotation's orbits are shaping. For example, for the horizontal edge, inner edge rotation gives orbit equal to $(1\ 2)$, but edge rotation gives orbit equal to $(3\ 4)$. Simpler speaking, we say that map has edge $(3\ 4)$ and inner edge $(1\ 2)$.

Further, let us assume that we have this combinatorial map given and try to picture as a graph's embedding in the plane:

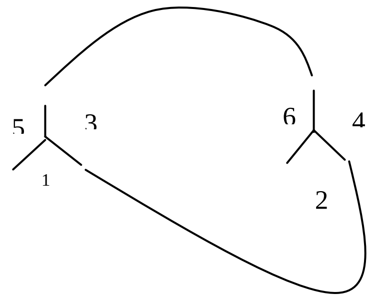
$$\begin{cases}(1\ 5\ 3)(2\ 6\ 4)\\(1\ 6\ 3\ 2\ 5\ 4)\end{cases}$$

Let us first picture vertex rotations as „halfedge rotations" clockwise in the plane:

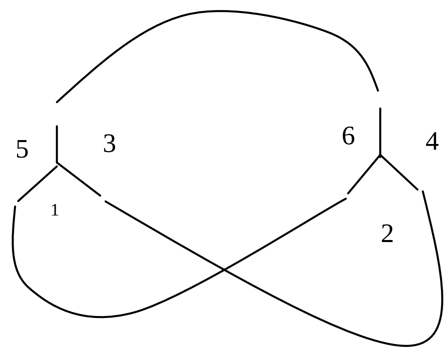
Let us go on with connecting halfedges into edges of the graph. First let us embed edge $(3\ 4)$ [and inner edge $(1\ 2)$]:



Next let come edge $(5\ 6)$ [inner edge $(3\ 4)$]:



At last let us embed edge $(1\ 2)$ [and inner edge $(5\ \ 6)$] and what we get?

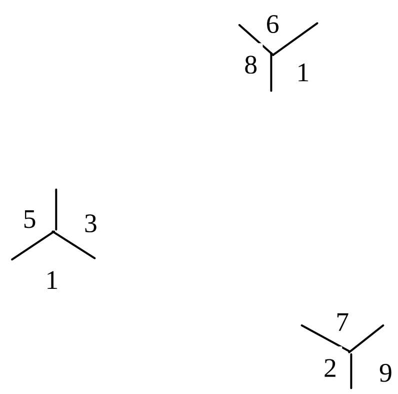


The drawing is done. It remains to find out that we did the same drawing where we started from above.

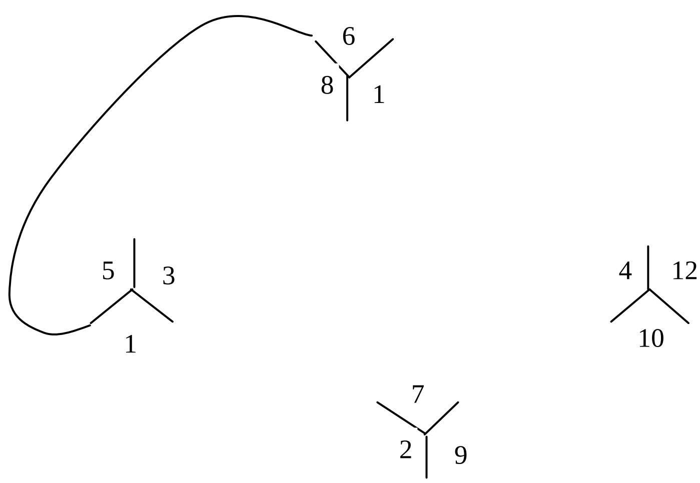
Let us next „draw the map", i.e., find corresponding graph's embedding in the plane:

$$\begin{cases}(1\ 5\ 3)(2\ \ 7\ \ 9)(4\ 12\ 10)(6\ 11\ 8) \\ \ (1\ 6\ 12\ 9)(2\ 8\ 5\ 4\ 11\ 7\ 10\ 3)\end{cases}.$$

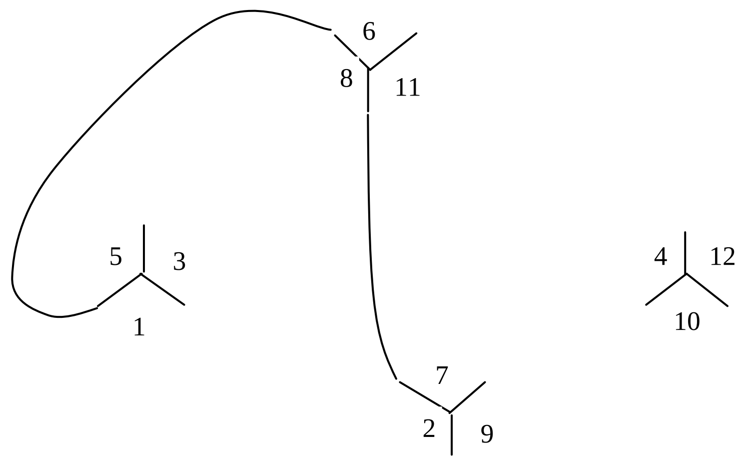
We start with halfedge's rotations:



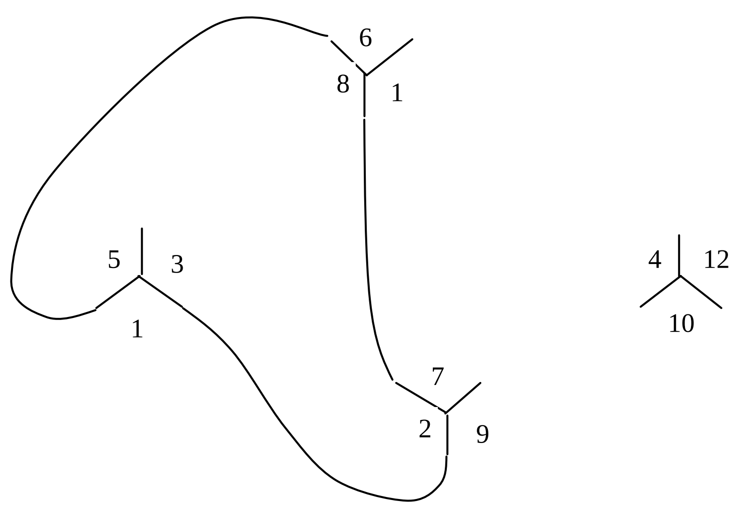
Let us embed edges in the order given: $(1\ \ 8),(2\ 11),(3\ 9),(4\ \ 6)(5\ 10),(7\ 12)$.



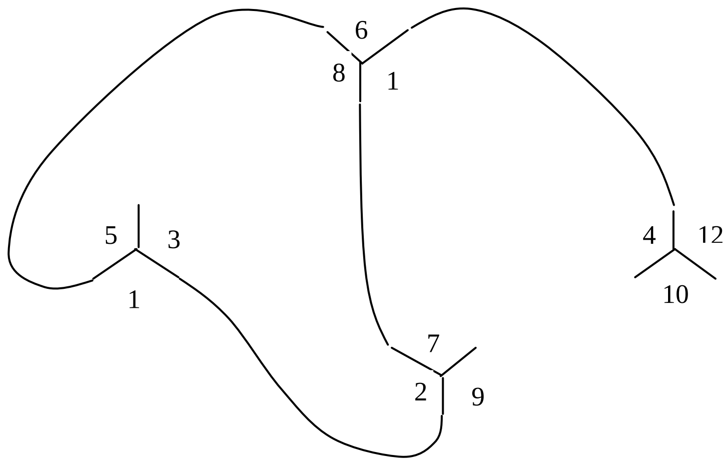
We embedded $(1\ \ 8)$. Remains edge rotation $(2\ 11),(3\ 9),(4\ \ 6)(5\ 10),(7\ 12)$.



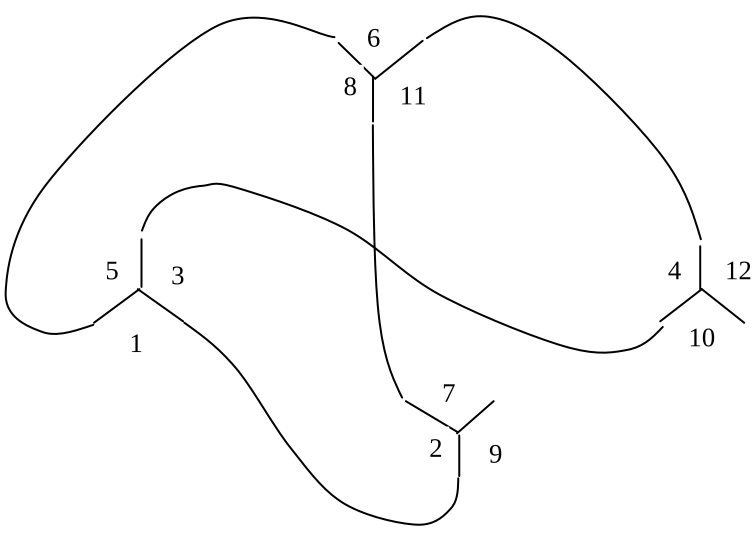
We embedded $(2 \ \ 11)$. Remains to be embedded edges from rotation $(3 \ 9), (4 \ \ 6)(5 \ 10), (7 \ 12)$.



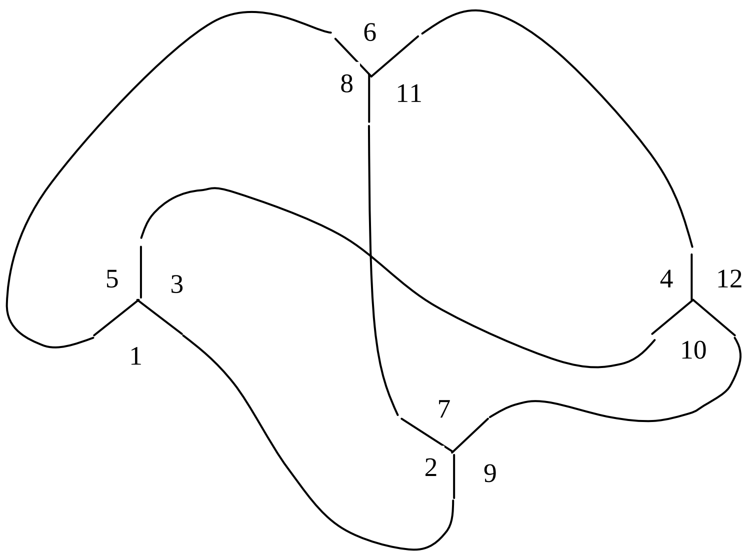
We embedded $(3 \ \ 9)$. Remains $(4 \ \ 6)(5 \ 10), (7 \ 12)$.



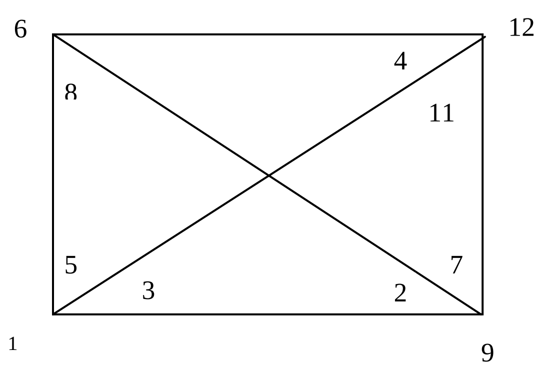
We embedded $(4\ \ 6)$. Remains $(5\ 10),(7\ 12)$.



We embedded $(5\ 10)$.Remains $(7\ 12)$.



Last edge is embedded. All halfedges are connected.We get a graph embedded in the plane. It is easy to see that more nice picture is possible:



Exercise: draw graph embedded in the plane corresponding to combinatorial map:

$$\begin{cases} (1\ 8\ 9)(2\ 5\ 3\ 6)(4\ 7\ 10) \\ (1\ 7\ 9\ 2\ 6)(3\ 5\ 4\ 8\ 10) \end{cases}.$$

Warning: for the first time a loop should appear as an edge in a looped graph.

Exercise: draw graph embedded in the plane corresponding to combinatorial map:

$$\begin{cases} (1\ 9\ 11)(4\ 12\ 8)(2\ 3\ 6)(5\ 7\ 10) \\ (1\ 10\ 6)(2\ 4\ 11)(3\ 5\ 8)(7\ 9\ 12) \end{cases}.$$

Hint: tetrahedron should be got.

Exercise: draw graph embedded in the plane corresponding to combinatorial map:

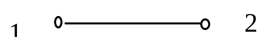
$$\begin{cases} (1\ 13\ 7)(2\ 10\ 11)(3\ 8\ 16)(4\ 17\ 9)(5\ 15\ 14)(6\ 12\ 18) \\ (1\ 14\ 6\ 11)(2\ 9\ 3\ 7)(4\ 18\ 5\ 16)(8\ 15\ 13)(10\ 12\ 17) \end{cases}.$$

Hint: prism graph should be got.

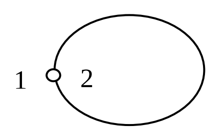
Simple combinatorial maps and their drawings

Identical permutation is denoted by e . Simplest map possible would be map with two corners e , which acts on C_2 . It is $\begin{cases} e \\ (1\ 2) \end{cases}$,

with corresponding drawing [one isolated edge]:



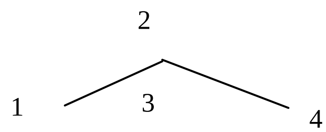
The dual map to it is $\begin{cases} (1\ 2) \\ e \end{cases}$ with its drawing

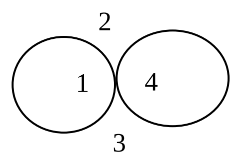


For identity that acts on C_{2m} , there corresponds graph with m isolated edges. Involution with m orbits action on C_{2m} has graph with m isolated loops. The only essentially empty graph would be that corresponding to C_0 , i.e., C_{2m} with $m = 0$. However, we better not specify cardinality of the universal set, but it is more convenient to assume that the cardinality m is some very large number $m = m_0$ that never is exceeded, and particular map may be specified with some chosen $m, m \le m_0$. Then empty graph would be graph without edges but having m_0 isolated edges, without specifying number of isolated edges. All graphs in the universal 'see' of edges would have $m = m_0$ edges. "Graph traditionally" would be connected component without isolated edges. The only inconvenience would be that we could not distinguish a component with only one edge.

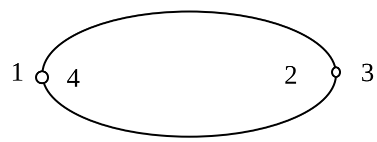
Maps on C_4

Map $\begin{cases} (2\ 3) \\ (1\ 2\ 4\ 3) \end{cases}$ with its drawing [and drawing of dual map]:

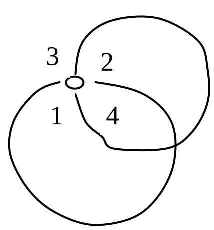




Selfdual map $\begin{cases} (1\ 4)(\ 2\ 3) \\ (1\ 3)(2\ 4) \end{cases}$ with its drawing:

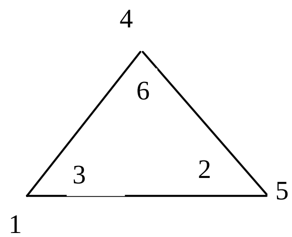


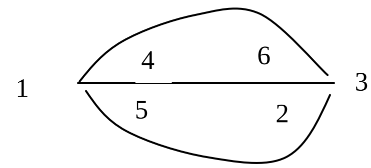
Selfdual map $\begin{cases} (1\ 3\ 2\ 4) \\ (1\ 4\ 2\ 3) \end{cases}$ with its drawing:



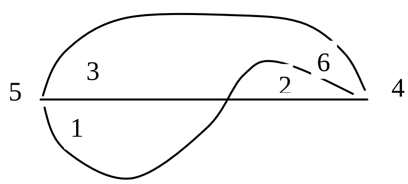
Maps on C_6

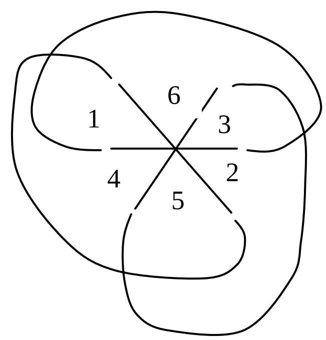
Map $\begin{cases}(1\ 3)(\ 2\ 5)(4\ 6)\\ (1\ 4\ 5)(2\ 6\ 3)\end{cases}$ with its drawing [and drawing of dual map]:



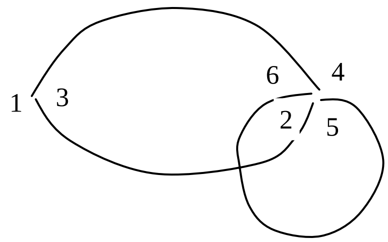


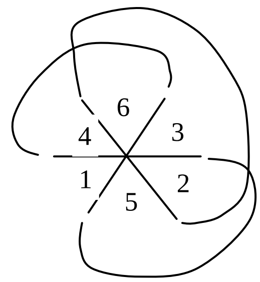
Map $\begin{cases}(1\ 5\ 3)(\ 2\ 6\ 4)\\ (1\ 6\ 3\ 2\ 5\ 4)\end{cases}$ with its drawing [and drawing of dual map]:





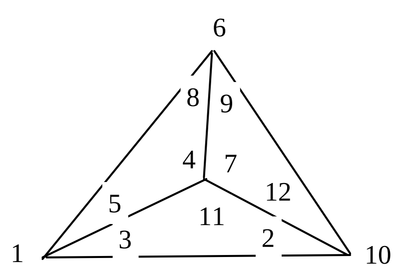
Map $\begin{cases}(1\ 3)(\ 2\ 6\ 4\ 5)\\ (1\ 4\ 6\ 3\ 2\ 5)\end{cases}$ with its drawing [and drawing of dual map]:





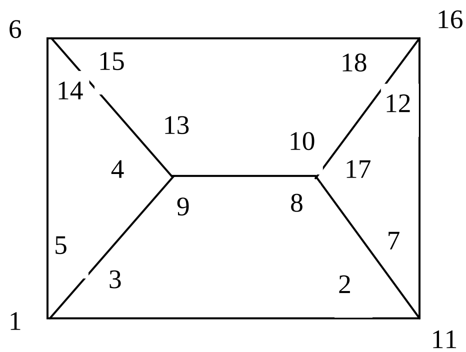
Other interesting maps

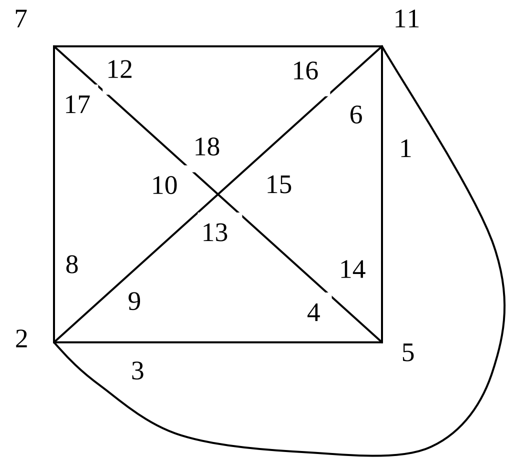
Tetrahedron – self dual map $\begin{cases}(1\ 5\ 3)(\ 2\ 12\ 10)(4\ 7\ 11)(6\ 9\ 8)\\ (1\ 6\ 10)(2\ 11\ 3)(4\ 8\ 5)(7\ 12\ 9)\end{cases}$ with its drawing:



Map of prizm graph $\begin{cases}(1\ 5\ 3)(\ 2\ 7\ 11)(4\ 13\ 9)(6\ 15\ 14)(8\ 10\ 17)(12\ 18\ 16)\\ (1\ 6\ 16\ 11)(2\ 8\ 9\ 3)(4\ 14\ 5)(7\ 12\ 17)(10\ 18\ 15\ 13)\end{cases}$ with its drawing

[and drawing of dual map]:





By the way we get to know that dual graph of prism graph is graph K_5^-, i.e., the full graph with five vertices with one edge missing.

Vertex split-merge operation

By multiplying permutation by transposition from left side in the permutation two orbits either merge into one or one becomes divided into two orbits depending on whether elements of transposition are in two distinct orbits or both in one orbit:

$$(a\ b) * Aa; Bb = AaBb$$

in the first case, when orbits merge or:

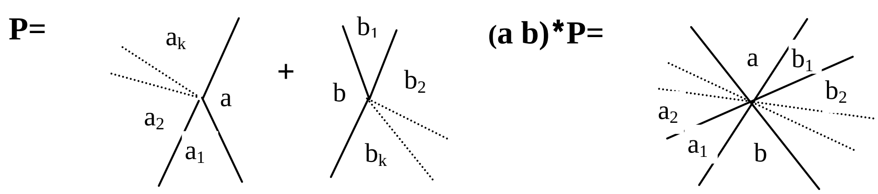
$$(a\ b) * Aa\,Bb = Aa; Bb$$

in the second case, when orbit is split into two new orbits,

or, using conjugate operator $\pm, \mp$:

$$(a\ b) * Aa \pm Bb = Aa \mp Bb\,.$$

Applying this operation to combinatorial map, its vertices are either merged or split. In the picture it is illustrated how it looks like geometrically. Let $A = a_1 a_2 ... a_{k_A}$ and $B = b_1 b_2 ... b_{k_B}$.



The result we get is equal to $AaBb = a_1 a_2 ... a_{k_A} a\, b_1 b_2 ... b_{k_B} b$. Let us take a note of the fact that in the formula indices would arrange more symmetrically if we chose to number corners not clockwise, but anticlockwise [with $(a\ b)$ considering as if standing before other corners].

This geometrical interpretation of multiplication of a single transposition gives an interesting graph-theoretical result.

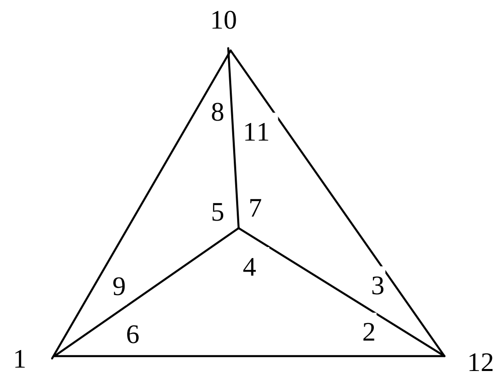
Let us call the operation corresponding to $(a\ b) \cdot P$ corner split-merge operation

Theorem: Producing corner split-merge operation with all pairs of corners from edge rotation we get the dual graph of the graph.

Proof of the fact is trivial from combinatorial point of view. But, graph-theoretically it gives an impression of some magic. Let us look in an example what goes on.

Tetrahedron $(1\ 9\ 6)(2\ 3\ 12)(4\ 5\ 7)(8\ 10\ 11)$ with face rotation

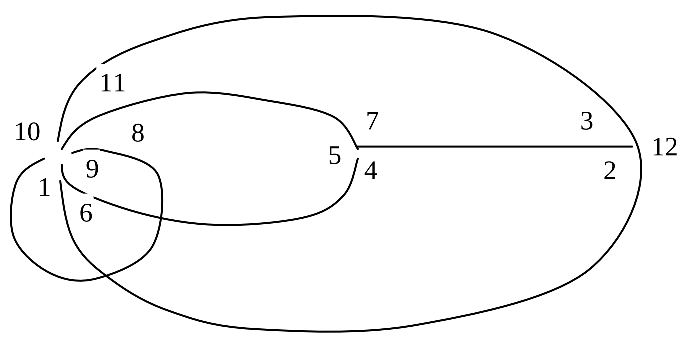
$(1\ 10\ 12)(2\ 4\ 6)(3\ 11\ 7)(5\ 8\ 9)$ and edge rotation $(1\ 8)(5\ 11)(2\ 7)(6\ 12)(3\ 10)(3\ 10)(4\ 9)$.



Let us apply to tetrahedron the operation of multiplying from the left transpositions from edge rotation.

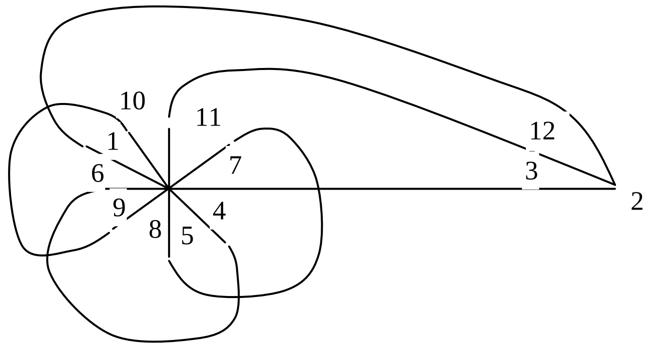
Let us multiply by first transposition from edge rotation:

$(1\ 8) * P = (9\ 6\ 1\ 10\ 11\ 8)(2\ 3\ 12)(4\ 5\ 7)$. We get:



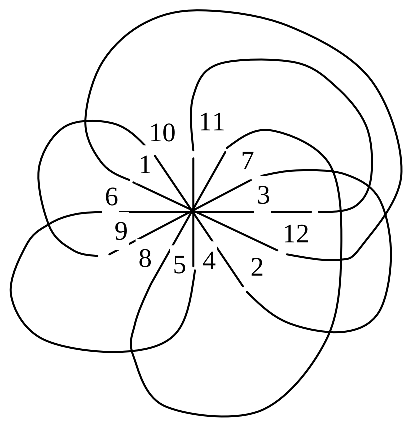
Let us multiply by second transposition from edge rotation:

$(5\ 11) * P = (7\ 4\ 5\ 8\ 9\ 6\ 1\ 10\ 11)(2\ 3\ 12)$. We get:



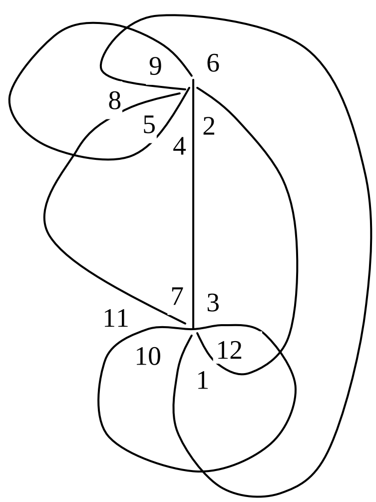
Let us multiply by third transposition from edge rotation:

$(2\ 7) * P = (3\ 12\ 2\ 4\ 5\ 8\ 9\ 6\ 1\ 10\ 11\ 7)$. We get:



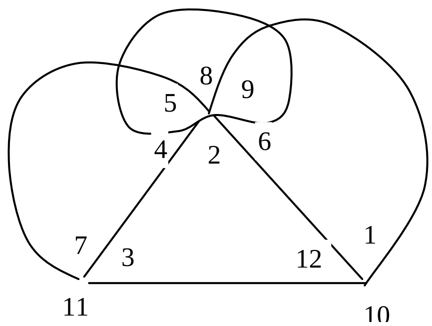
Let us multiply by fourth transposition from edge rotation:

$(6\ 12) * P = (2\ 4\ 5\ 8\ 9\ 6)(1\ 10\ 11\ 7\ 3\ 12)$. We get:



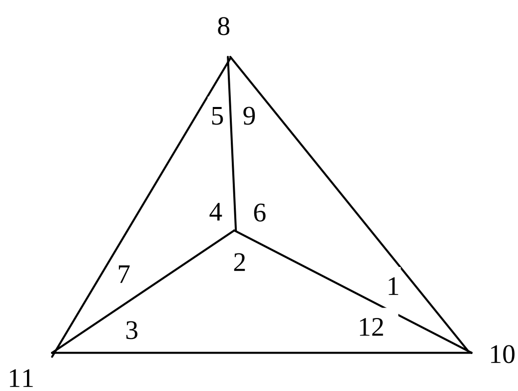
Let us multiply by fifth transposition from edge rotation:

$(3\,10)*P=(11\ 7\ 3)(12\ 1\ 10)(2\ 4\ 5\ 8\ 9\ 6)$. We get:



Let us multiply by sixth transposition from edge rotation:

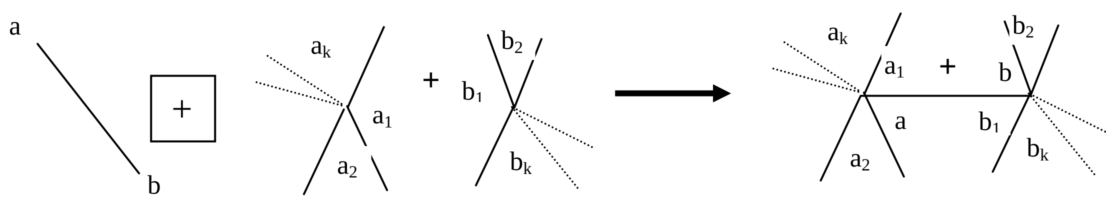
$(4\ 9)*P=(6\ 2\ 4)(5\ 8\ 9)(11\ 7\ 3)(12\ 1\ 10)$. We get:



This map is dual map to previous map, i.e., its vertex rotation is equal to face rotation of the previous map.

Joining of new edge.

Theorem 1: New edge with inner edge rotation $(a\ b)$ to map P is joined with respect to corner pair $(a_1\ b_1)$ by multiplication $(a\ a_1)(b\ b_1)\cdot P$.



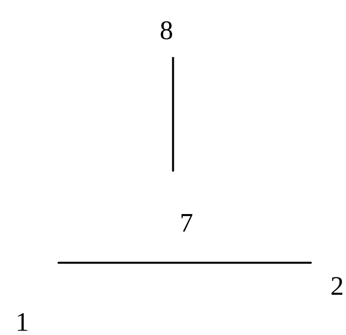
Proof:

$$(a\ a_1)(b\ b_1)\cdot P = (a\ a_1)(b\ b_1)(a_1\ a_2 ... a_{k_A})(b_1\ b_2 ... b_{k_B})P' = (a\, a_2\ ... a_{k_A}\ a_1)(b\, b_2 ... b_{k_B}\ b_1)P'$$

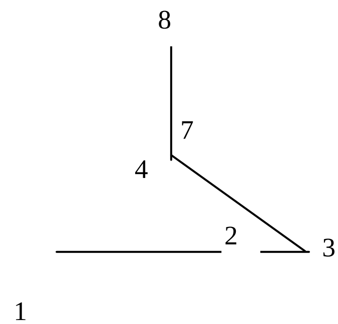
where P' is part of edge rotation which remains unchanged.

Example. Let us illustrate building of a graph by joining new edges.

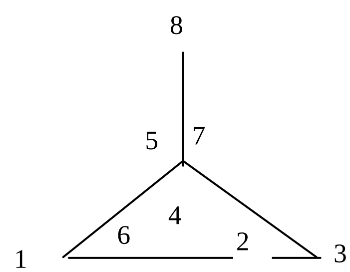
Let us assume that graph has already two isolated edges. $P = id$:



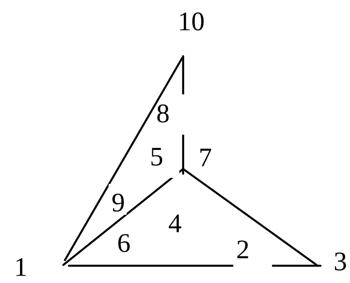
Let us add new (inner) edge $(3\ 4)$ with respect of corner pair $(2\ 7)$. We perform operation $(3\ 2)(4\ 7)\cdot id = (3\ 2)(4\ 7)$:



Let us add new (inner) edge $(5\ 6)$ with respect to corner pair [or non-edge] $(4\ 1)$. We perform operation $(5\ 4)(61)\cdot(3\ 2)(4\ 7) = (1\ 6)(2\ 3)(4\ 5\ 7)$:

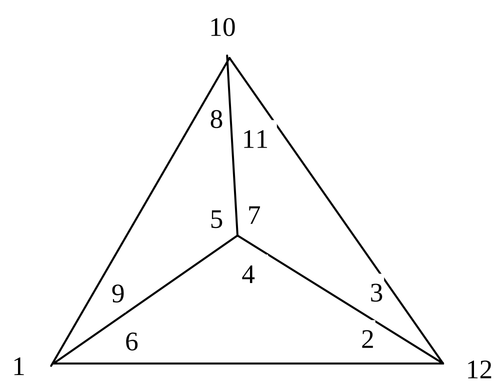


Let us add (inner) edge $(9\ 10)$ with respect to corner pair [non-edge] $(1\ 8)$. We perform operation $(9\ 1)(10\ 8)\cdot(1\ 6)(2\ 3)(4\ 5\ 7) = (1\ 9\ 6)(2\ 3)(4\ 5\ 7)(8\ 10)$:



Let us add new (inner) edge $(11\,12)$ with respect to corner pair $(10\ 3)$. We perform operation

$(11\,10)(12\ 3)\cdot(1\ 9\ 6)(2\ 3)(4\ 5\ 7)(8\ 10)=(1\ 9\ 6)(2\ 3\ 12)(4\ 5\ 7)(8\ 10\ 11)$:



Classes of combinatorial maps with fixed edge rotation.

For a given combinatorial map P its edge rotation is equal to $\rho = P\pi P^{-1} = \pi^{P^{-1}}$.

Let us remind that permutations α and β are called conjugated with respect to permutation P if $\alpha = \beta^{P}$. We see that in combinatorial map inner edge rotation and inner edge rotation are conjugated with respect to vertex rotation.

Are there other maps with the same edge rotation [within the class of maps with fixed inner edge rotation]? Yes, each permutation with respect which π and ρ are conjugated fits for vertex rotation of such map.

Let inner edge rotation π be fixed. All such classes form a class of maps K :

$$K = \{P \mid \pi_P = \pi\}.$$

Let us define class of combinatorial maps K_ρ that contain maps with fixed edge rotation ρ :

$$K_\rho = \{P \mid \pi_P = \pi \wedge \rho_P = \rho\}.$$

For different values of edge rotation ρ class K have subclasses K_ρ. Between these classes one class is special, namely, K_π that which has $\pi = \rho$, i.e., for the members of this class edge rotation and inner edge rotation coincided:

$$K_\pi = \{P \mid \pi_P = \pi \wedge \rho_P = \pi\}.$$

Maps of this class are called selfconjugate maps. Thus, K_π is the class of selfconjugate maps. This class is not empty; it contains map (id, π). Indeed, $\rho_{(id,\,\pi)} = \pi^{id} = \pi$. Thus, map with only isolated edges is example of selfconjugate maps. We shall see further other examples too.

In order to learn to consider maps of K_ρ, let us find what edge rotation has multiplication of two maps.

Theorem 1. For two maps $S, T \in K$ there holds: $\rho_{S\cdot T} = \rho_T^{S^{-1}}$.

Proof:

$$\rho_{S\cdot T} = S\cdot T\cdot \pi\cdot T^{-1}\cdot S^{-1} = S\cdot \rho_T\cdot S^{-1} = \rho_T^{S^{-1}} .$$

Let us define class where P is fixed:

$$P\cdot K_\sigma = \{P\cdot Q \mid Q \in K_\sigma\},$$

i.e., $P \cdot K_\sigma$ contain all maps from K_σ multiplied from left with a fixed map P. From theorem 1 we have that $P \cdot K_\sigma \subseteq K_\rho$, where $\rho = \sigma^{P^{-1}}$. Let us prove that equality holds. Let us first prove that K_π is a group.

Theorem 2 . K_π is a group.

Proof: Group operation is, of course, multiplication of permutations. Let us show that K_π is closed with respect to multiplication of permutations. If $\rho_S = \rho_T = \pi$ then $\rho_{S \cdot T} = S \cdot T \cdot \pi \cdot T^{-1} \cdot S^{-1} = S \cdot \pi \cdot S^{-1} = \pi$. Class is closed with respect to reversion operation too: $\rho_{S^{-1}} = S^{-1} \cdot \pi \cdot S = \pi$ .Class K_π contains a map that corresponds to identity permutation. Thus, all group's requirements K_π satisfies, and K_π is a group.

Let us note that K is class of maps with fixed π and in the same time it is a group [or isomorphic to] that is called symmetry group. [Mostly it is designated as S_n; in our case we use designation S_{2m}]. Of course, K_π is subgroup of group K. Let us prove coincidence of two classes.

Theorem 3. $P \cdot K_\sigma = K_{\sigma^{P^{-1}}}$. ($P \cdot K_{T^P} = K_T$.)

Proof: If $\sigma = \pi$, then $P \cdot K_\sigma$ is a left coset of K_π equal to $K_{\pi^{P-1}}$ and theorem is proved. Let $\sigma \neq \pi$ and Q is a map, that holds $Q \cdot K_\pi = K_\sigma$, i.e., K_σ is left coset of K_π and Q is one of elements of class K_σ: $\rho_Q = \pi^{Q^{-1}}$ and $Q \in K_\sigma$ according to theorem 1. If $P \cdot Q = R$, then holds

$$P \cdot K_\sigma = P \cdot Q \cdot K_\pi = R \cdot K_\pi = K_{\pi^{R^{-1}}} = K_{\sigma^{P^{-1}}}.$$

We used the fact that there holds $P \cdot (Q \cdot K_\pi) = (P \cdot Q) \cdot K_\pi$. Theorem is proved.

Besides the theorem, we got that class K_ρ is left coset of the group K_π in the group K. Let us fix this as corollary.

Corollary 1. In the group K left cosets to the subgroup K_π are classes with fixed edge rotation.

Thus, arbitrary combinatorial map P with edge rotation equal to $\rho = \pi^{P^{-1}}$ belongs to coset $K_\rho = P \cdot K_\pi$ to K_π.

Let us consider some properties of class K_π.

Lemma 1. $P^\pi = P$ holds iff $P \in K_\pi$.

Proof:

$$\rho_P = \pi \equiv \pi = P \cdot \pi \cdot P^{-1} \equiv P = \pi \cdot P \cdot \pi \equiv P = P^\pi .$$

Let c be orbit of P. We call orbit c^π selfconjugate with respect to orbit c (with respect to π) if $c = c^\pi$. If in a map P each orbit has its conjugate orbit (with respect to π) belonging to P or it is selfconjugate then it is called selfconjugate. From lemma 1 we have that K_π is the class of selfconjugate maps (with respect to π). Let us formulate this fact as theorem.

Theorem 4. The class of selfconjugate maps is equal to K_π.

Let us say that involution τ contains involution σ writing $\sigma \subseteq \tau$ if each transposition of σ is also transposition of τ. Let us clarify something about structure of selfconjugate maps.

Theorem 5. K_π (that is isomorphic to normal subgroup of S_{2m}) is isomorphic to group $S_m \cdot S_2^m$.

Proof: Let $P \in K_\pi$, and P as permutation acts on universal set of corners C, and $C_1 \cup C_2$ is subdivision of C that is induced by π. In that case there exists an involution $\sigma \subseteq \pi$, that $P = Q \cdot \sigma$ and in the same time Q orbits belong [as sets of elements] either to C_1 or C_2, i.e., if orbit c belongs to C_1, then c^π belongs C_2, or reversely. Q can be expressed as $Q_1 \cdot Q_2$, where Q_1 has corners belonging to C_1, and Q_2 has corners belonging to C_2. But in that case, Q_1 and Q_2 are isomorphic to each other and isomorphic to some permutation from S_m and $\sigma \subseteq \pi$ is isomorphic to some permutation from S_2^m, and P is isomorphic to permutation from $S_m \cdot S_2^m$. Theorem is proved.

How many there are selfconjugate maps?

Theorem 6. $\left|K_{\pi}\right| = m! \times 2^{m}$.

Proof: $\left|S_{m}\right| = m!$; $\left|S_{2}^{m}\right| = 2^{m}$.

How many there are edge rotations, i.e., how many left coset has the group K_{π} ?

Theorem 7. Group K_{π} has (itself including) $(2m-1)!!$ left cosets.

Proof: K_{π} (including itself) has as many left cosets as many edge rotations it is possible to generate, namely, $(2m-1)!!$. Indeed, there holds:

$$(2m-1)!! \times m! 2^{m} = (2m)!.$$

Combinatorial knot.

Let combinatorial map P be given as permutation P that acts on set of corners C and edge rotation is equal to $\rho = \pi^{P^{-1}}$. Let partition $C_1 \cup C_2$ on C is given such that it induces both π and ρ, i.e., for each edge and inner edge their ends belong both to C_1 and C_2. In that case we are saying that C is partitioned well or is colored in two colors well, or we say that $C_1 \cup C_2$ is well coloring of the universal set C that is induced by P.

Does such well colorings exist always and how many they are?

Theorem 1. Arbitrary map P always induces some well coloring of C.

Proof: Let $c_1c_2...c_{2k}$ be cyclical sequence of corners such that starting with arbitrary c_1, $c_2 = c_1^{\pi}$, and $c_3 = c_2^{\rho}$ and so on in alternating way, i.e., $c_{2i} = c_{2i-1}^{\pi}$ and $c_{2i+1} = c_{2i}^{\rho}$, for indices $i = 1,...,k$, in a way that, sequence closing $c_{2k}^{\rho} = c_{2k+1} = c_1$. Such cyclical sequence has even number of elements and always exists. Let for a moment assume number of elements being odd. In that case $c_{2k-1} = c_1^{\pi} = c_2$, and $c_{2k-2}^{\rho} = c_2$, and $c_{2k-2} = c_2^{\rho} = c_3$ and so on until $c_{k+1} = c_k$, but it is not possible.

In case not all corners are exhausted we go on starting from new non searched element. Let all corners would be exhausted by such cyclical sequence. Let us put odd elements in set C_1 and even elements in set C_2. We have got well coloring of set C that is induced by P.

Let us define permutation μ with orbits that was described in the previous case, i.e., if $C_1 \cup C_2$ is well coloring of set C then $c^{\mu} = c^{\pi}$, if $c \in C_1$, and $c^{\mu} = c^{\rho}$, if $c \in C_2$. Permutation μ we are going to call *combinatorial knot* of the map P. Further we shall see the motivation for this name. Frequently we would say simply knot in place of combinatorial knot. In graphical corner interpretation of combinatorial map combinatorial knot is really something similar to knot. P. Bonnington and Ch. Little are calling this combinatorial object zigzag walk.

How many combinatorial maps there are?

Lemma 1. If μ is knot of P then permutation μ', that is equal to μ with some orbit in opposite direction, is also a knot of this map P.

Proof follows directly verifying assertion.

But what is difference between knots μ and μ'? For the orbit that changed its direction corners change their coloring with respect to partition of C in C_1 and C_2.

Knot μ is determined only by π and ρ, i.e., it does not depend on particular map P. Thus μ is common for all the class K_ρ. K_ρ contains a map that corresponds to permutation μ, because $\rho_\mu = \rho$. Indeed, we get a theorem:

Theorem 2. $\pi^\mu = \rho$.

Proof:

$$(c\ c^\pi) \in \pi \Rightarrow (c\ c^\pi)^\mu = (c^\pi\ c^{\pi\rho})[\in \rho] \vee (c\ c^\pi)^\mu = (c^\rho\ c^{\pi\pi})[\in \rho].$$

Let us denote by $\mu(\pi,\ \rho)$ arbitrary knot induced by π and ρ, i.e., that is a knot for each map from K_ρ. We may deduce a theorem:

Theorem 3. $K_\rho = \mu(\pi,\ \rho) \times K_\pi$.

The proof follows directly from what was said previously. This result may be formulated in the following way:

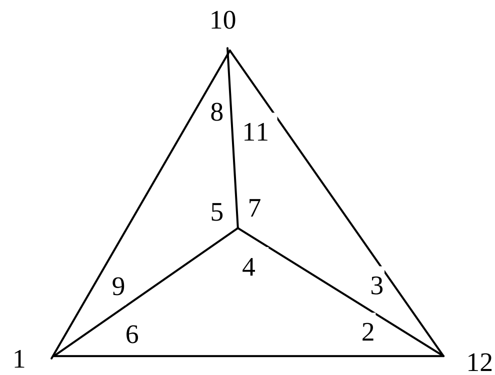
Corollary 1. Each combinatorial map may be expressed as multiplication of its knot with some selfconjugate map.

This selfconjugate map that is equal to $\mu^{-1} \cdot P$, we are going to call knotting and denote with letter α, thus corollary says that:

$$P = \mu \cdot \alpha.$$

Illustration of the combinatorial knot and knotting.

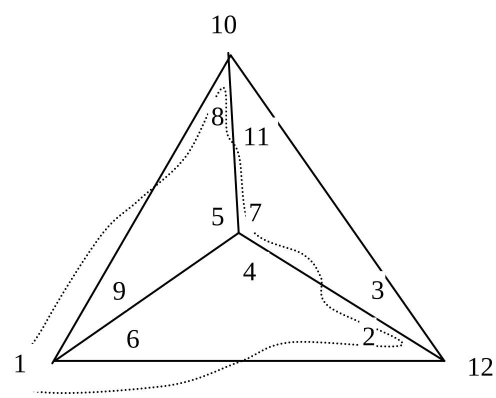
Let map tetrahedron be given with vertex rotation $(1\ 9\ 6)(2\ 3\ 12)(4\ 5\ 7)(8\ 10\ 11)$ and face rotation $(1\ 10\ 12)(2\ 4\ 6)(3\ 11\ 7)(5\ 8\ 9)$ and edge rotation $(1\ 8)(5\ 11)(2\ 7)(6\ 12)(3\ 10)(3\ 10)(4\ 9)$.



Let us consider algorithm of finding of combinatorial map following its definition. Let us start to construct μ with corner 1:

$$1^{\pi} \Rightarrow 2$$
$$2^{\rho} \Rightarrow 7$$
$$7^{\pi} \Rightarrow 8$$
$$8^{\rho} \Rightarrow 1$$

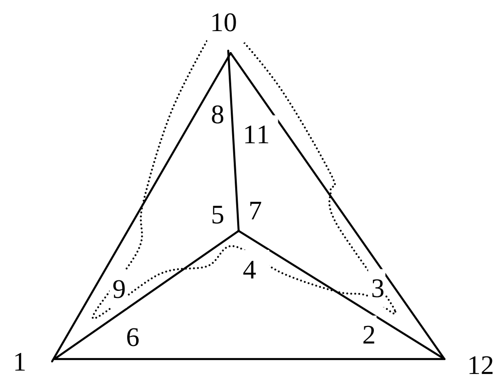
We have found one orbit of knot: $(1\ 2\ 7\ 8) \subset \mu$. Let us picture the found orbit of the knot in the picture of the graph:



Let us go on with new orbit starting with corner 3:

$$3^{\pi} \Rightarrow 4$$
$$4^{\rho} \Rightarrow 9$$
$$9^{\pi} \Rightarrow 10$$
$$10^{\rho} \Rightarrow 3$$

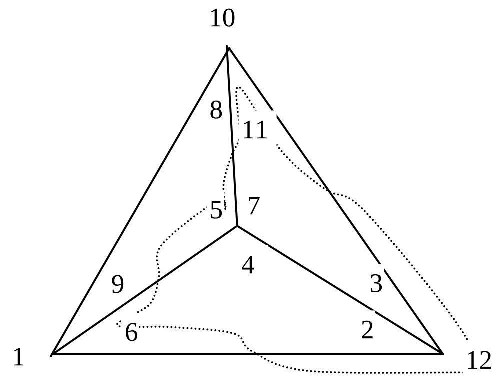
We have found one more orbit of the knot. We know that $(1\ 2\ 7\ 8)(3\ 4\ 9\ 10) \subset \mu$. Let us picture the found orbit of μ:



Let us find further next orbit starting with corner 5:

$$5^{\pi} \Rightarrow 6$$
$$6^{\rho} \Rightarrow 12$$
$$12^{\pi} \Rightarrow 11$$
$$11^{\rho} \Rightarrow 5$$

We have now found last orbit of the knot : $(1\ 2\ 7\ 8)(3\ 4\ 9\ 10)(5\ 6\ 12\ 11) = \mu$. Let us picture the found orbit:


10
8
11
5
7
4
9
3
2
6
1
12

Partial combinatorial maps

Combinatorial maps, that we considered up to now, were called geometrical, which have as their geometrical interpretation graph embeddings on surfaces.
Let us introduce a new notion – partial maps, that should have slightly modified and more general geometrical interpretation.
Let arbitrary pair of permutation be given (P, Q). We are calling this pair partial combinatorial map or shorter partial [sometimes p-map] map. Let us note that we do not specify any additional condition. As before, P is called partial map's vertex rotation and Q partial map's face rotation. As before, let us call multiplication of two permutations $Q \cdot P^{-1}$ partial map's edge rotation, denoting it R [or sometimes ρ], but now, in general, it should be distinct from involution. Orbits of R we are calling edges. Multiplication $P^{-1} \cdot Q$ we call inner edge rotation, as before, denoting it π.
It is convenient to give partial map as triple (P, Q, R) with condition to be holding $R = Q \cdot P^{-1}$.
It is easy to see that partial map (P, Q, R) is geometrical map in case R is involution without fixed elements. Is the requirement that R should be involution without fixed elements obligatory for geometrical map? It turns out that this should be decided by ourselves. But about this after we have learned about geometrical interpretation of partial maps.

Example 1: Partial map $((1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 2\ 3\ 4))$ has one vertex $(1\ 2\ 3\ 4)$, two faces $(1\ 3)$ and $(2\ 4)$, one edge $(1\ 2\ 3\ 4)$ that has inner edge $(1\ 2\ 3\ 4)$ in correspondence.

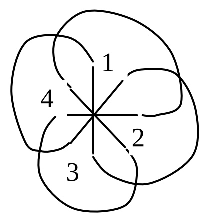
Example 2: Partial map $((1\ 2\ 3\ 4)(5\ 6\ 7), (1\ 5\ 3\ 7\ 2\ 6\ 4), (1\ 7)(2\ 5)(3\ 6))$ has two vertices $(1\ 2\ 3\ 4)$ and $(5\ 6\ 7)$, one face $(1\ 5\ 3\ 7\ 2\ 6\ 4)$, three edges $(1\ 7), (2\ 5), (3\ 6)$ of degree two and one of degree one (4) that have inner edges $(1), (2\ 5), (3\ 6), (4\ 7)$ in correspondence.

The drawing of partial map

Let us look how to draw partial map and after we are going to prove that this operation should be performable always. Let a partial map is given and $(c_1 ... c_k)$ is one of its vertices of order k. Let us draw in the plane star graph with $2k$ halfedges, putting labels for corners clockwise in every second corner. After all vertices are drawn in this way, let us connect halfedges, in order to draw edges of the graph, in the following way: in order to implement on “side” $(a\ b\ ...)$ of this edge, let us find corners with labels a and b, and connect these halfedges that go clockwise after these corners.

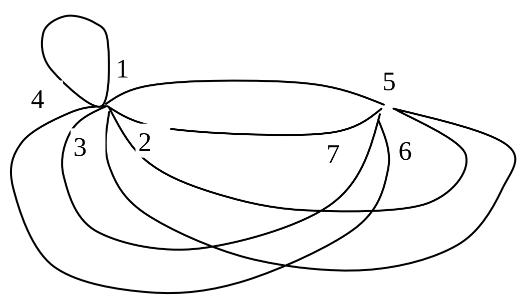
Let us persuade ourselves that drawing of the partial map of example 1 looks like this:

$$\begin{cases} (1\ 2\ 3\ 4) \\ \dfrac{(1\ 3)(2\ 4)}{(1\ 2\ 3\ 4)} \end{cases}$$



Let us consider the drawing of the partial map of example 2:

$$\begin{cases} (1\ 2\ 3\ 4)(5\ 6\ 7) \\ \dfrac{(1\ 5\ 3\ 7\ 2\ 6\ 4)}{(1\ 7)(2\ 5)(3\ 6)} \end{cases}$$



The image of partial map

In the previous chapter, when drawing of partial map was made, we labeled only every second corner, and half of corners remained without labels. In this chapter we are going to formalize the object that corresponds to the drawing of the partial map.

Let us assume that permutations P, Q, R are acting in the universal set C of corners, and $\overline{C}$ is a new set such that $|C| = |\overline{C}|$ and $C \cap \overline{C} = \varnothing$,with bijection $u : C \to \overline{C}$ given. For $c \in C$, let us denote $u(c)$ by $\overline{c}$, and succession of elements $c\overline{c}$ by $\tilde{c}$. For $u : C \to \overline{C}$ reverse bijection $\overline{u} : \overline{C} \to C$ is defined that $\overline{u}(\overline{c}) = c$ holds. Let us use the same denotations for orbits of permutations and for whole permutations too: if we have orbit $c = (c_1 ... c_k)$, then [induced by u]

$\bar c = (\bar c_1 ... \bar c_k)$, and $\tilde c = (c_1 \bar c_1 ... c_k \bar c_k)$. Similarly for permutations: if P acts in the set C, then [induced by u] $\bar P$ acts in the set $\bar C$.
Further, making practical drawings, we shall use convention to put corner $\bar c$ after corner c in clockwise direction.
Let us now show that drawing of the partial map has a combinatorial map in correspondence that we shall call image of the partial map.

Theorem 1. Let (P, Q, R) be arbitrary partial map. Then partial map $(\tilde P, Q \cdot R^{-1})$ is geometrical combinatorial map.

Proof: Let us denote p-map $(\tilde P, Q \cdot R^{-1})$ that corresponds to p-map (P, Q, R) by (P, Q). In order to prove assertion of the theorem, we must prove that in the p-map (P, Q) all edges are of degree two. Let us take arbitrary element $c \in C$. Applying for this element $Q\bar R^{-1}$, we get c^Q, because $\bar R^{-1}$ do not act on elements from set C and leave this element $c \in C$ on place.
Applying for element c^Q permutation $(\tilde P)^{-1}$, we get $\bar c^{\bar Q \cdot \bar P^{-1}}$, because $u : c^Q \mapsto \bar c^{\bar Q}$. Starting from element $c \in C$, first two elements of the edge are $(c\ \bar c^{\bar Q \bar P^{-1}} ...)$. Further, applying $Q\bar R^{-1}$ once more, we get $\bar c$, because $\bar Q \bar P^{-1} \cdot \bar R^{-1} = \bar Q \bar P^{-1} \cdot (\bar Q \bar P^{-1})^{-1} = id$. At last, applying for $\bar c$ permutation $(\tilde P)^{-1}$, we get c. From this sequence of judgments results that arbitrary edge that started in C, has degree 2. The same should be done for an edge that starts in $\bar C$. Let $\bar c \in \bar C$ for some $c \in C$: similarly as before:

$$\bar c \mapsto c^{Q(\bar Q \bar P^{-1})^{-1}} = \bar c^{\bar P \bar Q^{-1}} \mapsto \bar c^{\bar P \bar Q^{-1} \cdot \tilde P^{-1}} = c^{PQ^{-1}} \mapsto c^{PQ^{-1} \cdot Q(\bar Q \bar P^{-1})^{-1}} = c^P \mapsto c^{P \cdot \tilde P^{-1}} = \bar c .$$

This edge that started in the set $\bar C$ has degree 2 too. Theorem is proved.

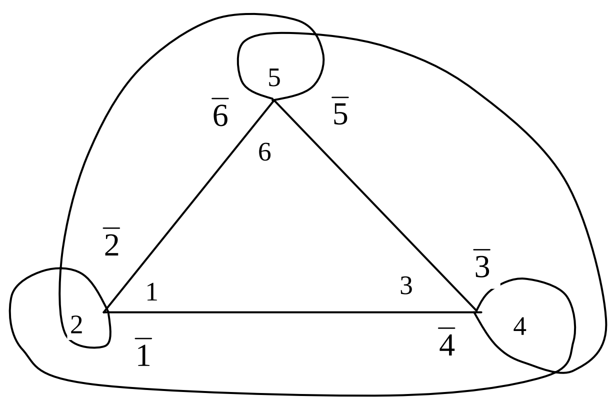
This theorem gives us effective tool. Geometrical map, that is to be connected with partial map, is isomorphic to partial map's drawing. Because of this fact we call this map c-map that corresponds to given p-map. c-map (P, Q) may be called partial map's image or combinatorial image, in order not to confuse it with graphical image.
Let us look on this in an example. Let p-map be given:

$$(P,\ Q,R)=\begin{cases}(1\ 2)(3\ 4)(5\ 6)\\ \underline{(1\ 3\ 5)(2\ 4\ 6)}\\ (1\ 4\ 5\ 2\ 3\ 6)\end{cases}.$$

This p-map has its c-map $(\mathsf{P},\ \mathsf{Q})$ in correspondence as follows:

$$(\mathsf{P},\ \mathsf{Q})=\begin{cases}(1\ \overline{1}\ 2\ \overline{2})(3\ \overline{3}\ 4\ \overline{4})(5\ \overline{5}\ 6\ \overline{6})\\ \underline{(1\ 3\ 5)(2\ 4\ 6)(\overline{6}\ \overline{3}\ \overline{2}\ \overline{5}\ \overline{4}\ \overline{1})}\\ (1\ \overline{4})(2\ \overline{3})(3\ \overline{6})(4\ \overline{5})(5\ \overline{2})(6\ \overline{1})\end{cases}.$$

Let us draw combinatorial image of the partial map in the plane:



Let find for example 1 c-map too:

$$(P,Q)=\begin{cases}(1\ \overline{1}\ 2\ \overline{2}\ 3\ \overline{3}\ 4\ \overline{4})\\ \underline{(1\ 3)(2\ 4)(\overline{1}\ \overline{4}\ \overline{3}\ \overline{2})}\\ (1\ \overline{2})(2\ \overline{3})(3\ \overline{4})(4\ \overline{1})\end{cases}$$

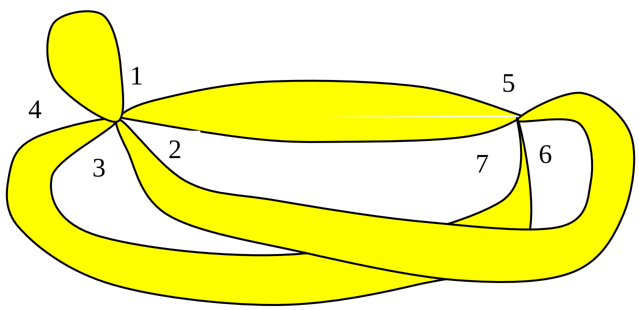
Pasting of edges in the partial map

Let us consider how to interpret combinatorial image of the partial map. Let us name faces of the combinatorial image that are not faces of the partial map, cut-out faces or black faces.

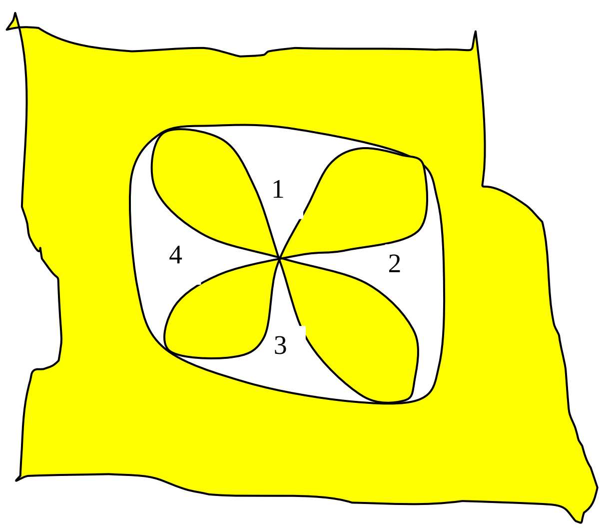
Let us say that cut-out face belongs to *halfedge* type if its degree is one. Let us say that it belongs to *edge* type if its degree is two. Let us say that it belongs to *essential* type if its degree exceeds two.

Let we have partial map with its c-map given. Let us cut out these edges from partial map that we called black or cut-out. We get object that by right can be called partial map. If we choose reverse operation to glue cut out edges back in the partial map, we get its image.

Here p-map of example 2 with cut-out faces.



More complicate situation is with example 1 p-map. Its image should be placed on genus two surface. In plane we could be comforted with picture [with crossings] below imagining that inner petals are turned upside down:



Shorter image of partial map

Let us drop from image edge type black faces, i.e., these black edges with degree two. Such picturing of partial map does not loose any information, i.e., from such image it is possible always to restore correct partial map. That means that fro $\overline{C}$ should be eliminated corners that belong to edges of degree two, i.e., orbits of degree two should not appear in $\overline{C}$. Such image we call shorter image of partial map.

Submaps

We may give another interpretation to partial map than we did before. We may consider as submap of other map.

Let us assume that p acts on set C, and S is subset of C. We denote by $p|_S$ restriction of permutation p on subset S, i.e., leaving only those elements of p that belong to subset S. If permutation p is given in cyclical form then $p|_S$ we get simply by elimination elements that do not belong to S.

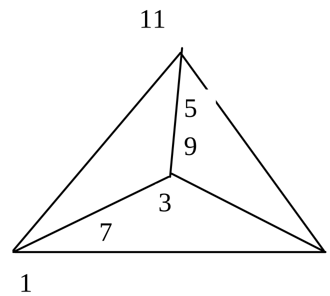
If we have partial map (P, Q) we may restrict both permutations at the same time, writing $(P, Q)|_S$. We say that partial map (P_1, Q_2) that is equal to $(P, Q)|_S$ is submap of the map (P, Q) i.e., its restriction on S.

Theorem. Each p-map is submap of arbitrary many p-maps [and c-maps too].

Proof follows from the fact that each p-map is submap of its image or c-map. Further, image can be taken as p-map that has its image and so on.

Further follows simple example of submap.

$C-$map	*submap*
$\left\{\begin{array}{c}(1\,12\;7)(2\;4\;6)(3\;8\;9)(5\,10\,11)\\(1\,11\;6)(2\;3\;7)(4\;5\;9)(8\,10\,12)\\\hline(1\,10)(2\;9)(3\,12)(4\,11)(5\;8)(6\;7)\end{array}\right.$	$\left\{\begin{array}{c}(1\;7)(3\;9)(5\,11)\\(1\,11)(3\;7)(5\;9)\\\hline(1\;5\;3)(7\;9\,11)\end{array}\right.$

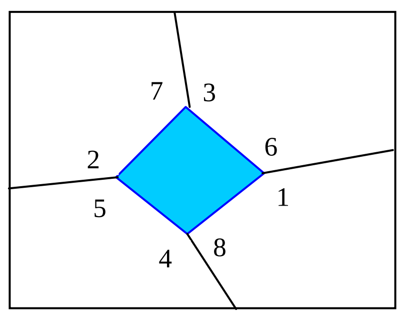


In this case partial map is submap of a c-map that is not its image.

Examples of partial maps

K_4 on torus with one face cut out:

$$\left\{\begin{array}{c}(1\;6)(2\;5)(3\;7)(4\;8)\\(1\;8\;3\;6\;2\;7\;4\;5)\\\hline(1\;4\;2\;3\;)(5\;6)(7\;8)\end{array}\right.$$



Theory of cycle covers. Uncolored covers

Searching graph embedding on topological surfaces, useful objects are graph cycles and cuts. For graph embeddings, graph cycles and dual objects cuts do the same what in traditional topology do closed lines. Here, speaking in language of combinatorics, we are dealing with graph cycles and cut, namely, we are going to use graph theoretical terminology for our combinatorial map theory.

The choice operator

Let us denote by $\because$ choice operation, that $P \because Q$ denote nondeterministic choice between P and Q, namely $c^{P \because Q}$ is equal either to c^P or c^Q. $P \because Q$ as permutation denotes such permutation, that is obtained applying nondeterministically either P or Q. Of course, its is question per se whether such permutation exists. We should see it later. Further, formally considering:

$$P \because Q = \{c_1 \mapsto c_2 | c_1, c_2 \in C \wedge (c_2 = c_1^P \vee c_2 = c_1^Q)\}.$$

For two given permutations P and Q always exists nonempty set $\{R | R = P \because Q\}$ that contains all possible permutations we get using choice operator, i.e., where P and Q are applied by nondeterministic choice. This set is not empty because it contains at least both these sets P and Q.

Walks, paths and cycles

For two arbitrary permutations P and Q sequence $a_1,...,a_n$ $(n > 0)$ is p-map's (P, Q) walk, if $a_{i+1} = a_i^{P \because Q}$ for all $(0 < i < n)$. Walk contains in general repetitions. If repetitions in walk are absent then it is called path. If path is closed then it is called cycle. It is easy to see that walk does not have in general permutation in correspondence. Does every cycle have permutation in correspondence? Let $\tau \in P \because Q$. Orbits of permutation τ are clearly cycles. Does there exist other cycles that are not orbits of $\tau \in P \because Q$? It is easy to see that no. But we formulate stronger assertion in theorem:

Theorem 1. Set $\bigcup_{\tau \in P \because Q} \bigcup_{a \in \tau} \{a\}$ $(= S)$ contains all cycles of (P, Q) and nothing else.

Proof. By induction assertion is correct at $|C| = 1$, and let us assume that it is correct for $|C| < 2n$. Let c be arbitrary cycle so that $|c| < 2n$ that is built using choice operator $\because$. By induction assertion it belongs both to cycles of (P, Q) and S. Let us consider restriction of permutations P and Q on set $C - |c|$. But other assertion is true also: by induction is true what theorem says about constriction of $P \because Q$ on $C - |c|$. Remains case when cycle has length $n+1$ and it is the only cycle of (P, Q), and we must persuade ourselves that it belongs to S always.

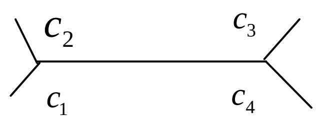
Cycle covers

For an arbitrary partial map (P, Q) elements of set $P \because Q$ we call cycle covers of (P, Q). Each such cover $\tau \in P \because Q$ is permutation, orbits of which are one or more cycles. This fact motivates the name of τ – cycle cover, i.e., each instance of $P \because Q$ cover some cycles of partial map so that the corresponding permutation covers all corners of partial maps. For some fixed cycle cover of partial map we write (P, Q, τ), i.e., (P, Q, τ) is partial map with fixed cycle cover τ. From the previous assertion follows:

Theorem 2. For each partial map's cycle a there exists some cover of cycles τ so that $a \in \tau$.

Cycle covers of c-maps

Let us consider further cycle covers only for geometrical maps, i.e., for which edge rotation is involution without fixed elements. If cycle cover for c-map is fixed then it is possible to distinguish four types of edges. First, let us denote corners for a fixed edge by (c_1, c_2, c_3, c_4), so that $c_2^P = c_1$, $c_3^\pi = c_1$ and $c_4^\rho = c_2$ (see picture below). We may memorize these indices in the following way: $\begin{array}{c|c} c_2 & c_3 \\ \hline c_1 & c_4 \end{array}$. By this convention (c_1, c_3) as always inner edge orbit and (c_2, c_4) edge orbit for corresponding graphical edge.



Next theorem describes four possible edge types in c-map if cycle cover is fixed.

Theorem 2. There are four possible cases for edge:

1) c_1 and c_2 belong to one cycle, and c_3 and c_4 belong to other cycle;

2) c_1 and c_4 belong to one cycle, and c_2 and c_3 belong to other cycle;

3) all corners belong to one cycle, but c_1 follows c_2 in the cycle and c_3 follows c_4 in the cycle;

4) all corners belong to one cycle, but c_1 follows c_4 in the cycle and c_3 follows c_2 in the cycle.

It is easy to see that there are excluded possibilities that corners of edge go into more than two cycles, and that the cycle crosses edge by diagonal, namely, cycle contains c_2 and c_4 (c_1 and c_3) without going through c_1 or c_3 (c_2 or c_4).

We are going to name edges according their type. Let first say that edge is *inner* edge if its corners belong to one cycle, and edge is *outer* edge if its corners belong to two cycles.

Further go names for each edge type:

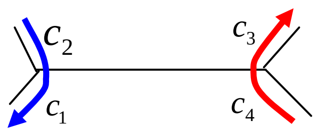
If $\kappa \in \tau$ and $c_1 = c_2^{\kappa}$ and $c_3 = c_4^{\kappa}$, then edge calls *cross edge* [3-rd case].

If $\kappa \in \tau$ and $c_1 = c_4^{\kappa}$ and $c_3 = c_2^{\kappa}$, then edge calls *recurrence edge* [4-th case].

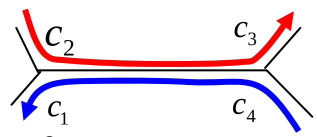
If $\kappa_1, \kappa_2 \in \tau$ are two distinct cycles, so that $c_1 = c_2^{\kappa_1}$ and $c_3 = c_4^{\kappa_2}$, then edge calls *cut edge* [1-st case].

If $\kappa_1, \kappa_2 \in \tau$ are two distinct cycles, so that $c_1 = c_4^{\kappa_1}$ and $c_3 = c_2^{\kappa_2}$, then edge calls *cycle edge* [2-nd case].

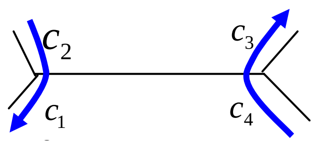
See below in picture:



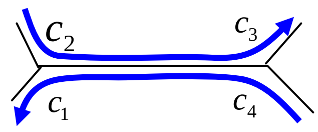
cut edge



cycle edge



cross edge



recurrence edge

Thus, inner edges are cross edge and recurrence edge, but outer edges are cycle edge end cut edge.

Map with fixed cycle coloring (P, Q, τ) induces edge type partitioning, and this may be reflected in partitioning of orbits in inner edge rotation too:

$$\pi = \pi_{cycle} \cdot \pi_{cut} \cdot \pi_{cross} \cdot \pi_{recurr},$$

where π_{cycle} contains orbits of type (c_1, c_3) , where (c_1, c_3) goes into cycle edge and so on.

This same partitioning of π may be induced on set C :

$$C = C_{cycle} \cup C_{cut} \cup C_{cross} \cup C_{recurr},$$

where C_{cycle} has corners of cycle [inner] edges, i.e. $C_{cycle} = C\big|_{\pi_{cycle}}$, and so on.

It may be convenient to use following self-evident designations too: $\pi = \pi_{inner} \cdot \pi_{outer}$ and $C = C_{inner} \cup C_{outer}$.

Let us consider examples of cycle covers. Let us have normal c-map $P = (1\ 8\ 9)(2\ 5\ 3\ 6)(4\ 7\ 10)$ with face rotation $Q = (1\ 7\ 9\ 2\ 6)(3\ 5\ 4\ 8\ 10)$. [Verify it!] Let us fix cycle cover $\tau = (1\ 8\ 10\ 4\ 7\ 9)(2\ 6)(3\ 5)$. In fig. 1. we see cycle cover τ , with cycle $(1\ 8\ 10\ 4\ 7\ 9)$ in blue color, cycle $(2\ 6)$ in red color, and cycle $(3\ 5)$ in green color. This cycle cover induces following edge types:

$\pi_{cut} = (1\ 2)(3\ 4)$ -- cut edges [red color];

$\pi_{cycle} = (5\ 6)$ -- cycle edges [green color];

$\pi_{cross} = (7\ 8)$ -- cross edge [yellow color];

$\pi_{recurr} = (9\ 10)$ -- recurrence edge [blue color].

Inner edges [*cycle* green and *cut* red] are marked with thicker lines.

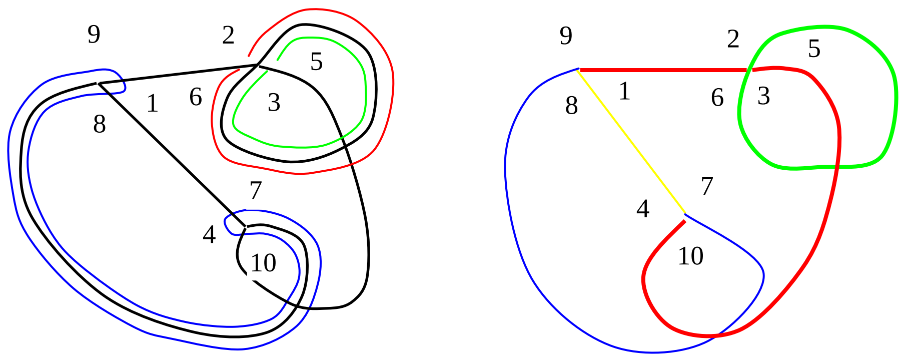


Fig. 1

In fig. 2 graph is shown, which vertices are cycles of cycle cover τ, and which edges appear between cycles or loops on cycle, if in (P, Q, τ) is edge that is incident with these cycles or that cycle. In this graph, inner edges of map induce edges and outer edges induce loops.

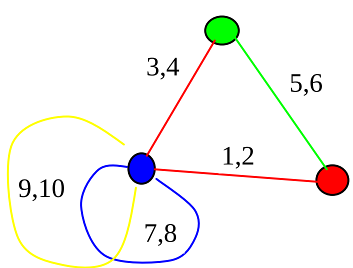
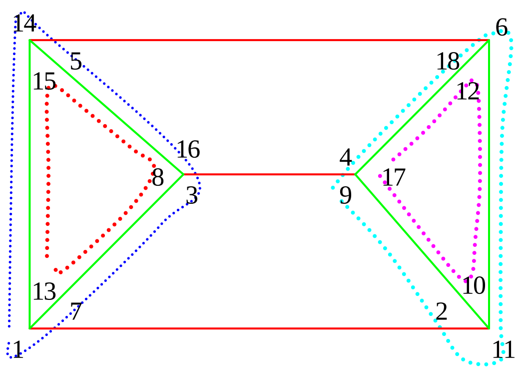
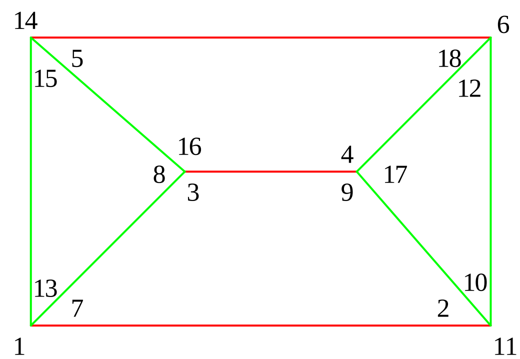


Fig. 2

Maps with only outer edges

Let us consider maps with fixed cycle cover that induces only outer edges. Let us see an example Prism [normal] map $P = (1\,13\,7)(2\,10\,11)(3\,8\,16)(4\,17\,9)(5\,15\,14)(6\,12\,18)$ with [induced] face rotation $Q = (1\,14\,6\,11)(2\,9\,3\,7)(4\,18\,5\,16)(8\,15\,13)(10\,12\,17)$ and fixed cycle cover $(1\,14\,5\,16\,3\,7)(2\,9\,4\,18\,6\,11)(8\,15\,13)(10\,12\,7)$ is given. We see that cycle cover induces only outer edges.





Types of edges and duality

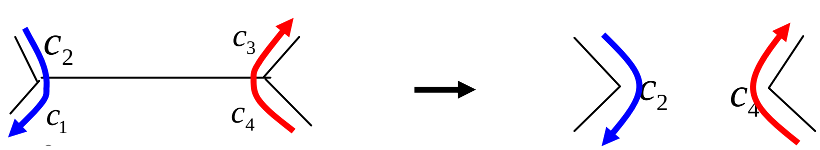
Let us have c-map with fixed cycle cover (P, Q, τ). It is easy to see, that exists dual map with the same cycle cover (Q, P, τ), what follows from definition of choice operator $\because$. Theorem follows:

Theorem. In the dual map with the same fixed cycle cover, for the cycle edges, there correspond cut edges and reversely, and for recurrence edges – cross edges and reversely; for inner edges correspond inner edges, and the same for outer edges.

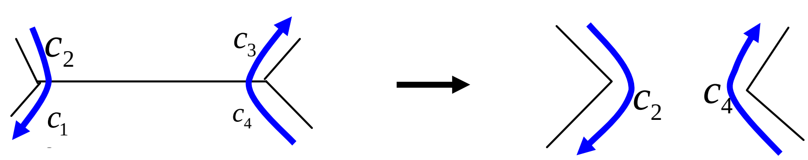
Cycle cover submap of combinatorial map

Cycles and cuts are essential graph invariants, so it is interesting as much as possible to acquire analogous notions in combinatorial outlook too.
For map $(P,\, Q,\, \tau)$ and dual map $(Q,\, P\ \tau)$ let us take cycle $\kappa \in \tau$. Let us denote restriction of permutation $\kappa\big|_{C_{cycle}}$ simply as κ_{cycle}. Similarly let us denote restriction of permutation $\kappa\big|_{C_{cut}}$ as κ_{cut}.
Let $(P,\, Q,\, \tau)$ be given and consider restriction of P on set $C_{cycle} \cup C_{recurr}$, denoting it $P_{cyclical}$. We call $P_{cyclical}$ cyclical part of vertex rotation, of course, with respect to fixed cycle cover τ. Similarly we consider restriction of τ on $C_{cycle} \cup C_{recurr}$, denoting it by $\tau_{cyclical}$, and calling it cyclical part of cycle cover. Let us consider partial map $(P_{cyclical},\, \tau_{cyclical})$, calling it maps [with fixed cycle cover] $(P,\, Q,\, \tau)$ cycle cover submap. Let us prove theorem:

Theorem. Cycle cover submap is geometrical map.
Proof. Geometrically eliminating cut or cross edge, as we see below, is equivalent with edge elimination with cycle cover remaining the same except to elements c_1, c_3 being dropped. But c_1, c_3 are elements of the eliminated inner edge.



cut edge



cross edge

Further we may have use of submap that correspond to some cyclical part of cycle cover.

Let τ be cycle cover of (P, Q) and $\sigma \subset \tau$, where σ contains one or more cycles from τ. Let C_σ contain corners in σ, i.e., $C_\sigma = C\big|_\sigma$. Let us consider p-map $(P\big|_{C_\sigma}, \sigma)$, what we call cyclic submap of map. Cyclic submaps of map are not geometrical maps in general.

Theorem. Let orbit κ contains recurrence edge. This edge is either bridge in cycle κ submap or submap is not planar.

Let us consider example of cycle cover submap. Let us have prism graph

$P = (1\,13\ 7)(2\ 10\ 11)(3\ 8\ 16)(4\ 17\ 9)(5\ 15\ 14)(6\ 12\ 18)$ with cycle cover equal with face rotation $(1\,14\ 5\ 16\ 3\ 7)(2\ 9\ 4\ 18\ 6\ 11)(8\ 15\ 13)(10\ 12\ 7)$. Noncyclical edges have corners $\{1, 2, 3, 4, 5, 6\}$, other corners go into cyclical edges. We obtain:

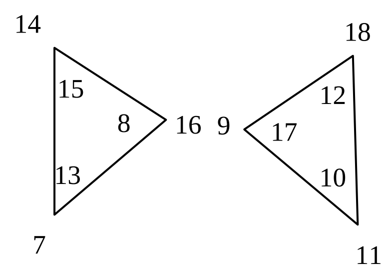
$P_{cyclical} = (7\ 13)(8\ 16)(14\ 15)(9\ 17)(10\ 11)(12\ 18)$;

$Q_{cyclical} = (7\ 9)(11\,14)(16\ 18)(8\ 15\ 13)(10\ 12\ 17)$;

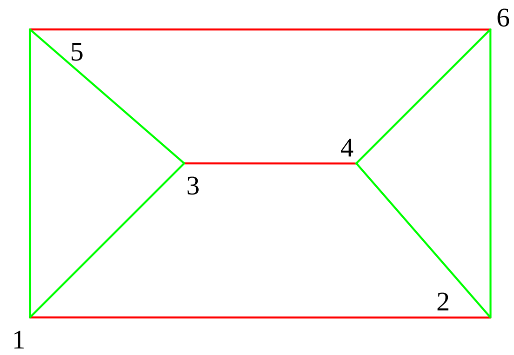
$\tau_{cyclical} = (7\ 14\ 16)(9\ 18\ 11)(8\ 15\ 13)(10\ 12\ 17)$.

What is cycle cover submap equal? We get it restricting P on cyclical edges corners:

$$\boxed{\left\{\begin{array}{c}(7\ 13)(8\ 16)(14\ 15)(9\ 17)(10\ 11)(12\ 18)\\ (7\ 14\ 16)(9\ 18\ 11)(8\ 15\ 13)(10\ 12\ 17)\\ \hline (7\ 15)(8\ 14)(9\ 12)(10\ 18)(11\ 17)(13\ 16)\end{array}\right.}$$



What remains is p-map $(P, Q)\big|_{C_{noncyclical}}$ what in picture below is shown as initial c-map's P submap:



Calculation of cycle cover and its characteristics

Let us persuade that for given $(P,\ Q,\ \tau)$ expression holds: $P \cdot \tau^{-1} = \rho_{cycle} \cdot \rho_{recurr}$. The same we get proving that $\rho_{cyclical} \cdot P = \tau$. For cyclical edge, τ we get as if using expression $\tau = \rho \cdot P$, but for cut edge or cross edge – as if using $\tau = P$; or , in other words, for cyclical edge we apply $\tau = Q$, but for noncyclical – $\tau = P$, i.e., that what to choice operator, namely, noncrossing edge it takes Q , but crossing edge it takes P .

Symmetric expression holds too: $P^{-1} \cdot \tau = \pi_{cycle} \cdot \pi_{recurr}$.

Theorem. For map with fixed cycle cover $(P,\ Q,\ \tau)$ we have in correspondence geometrical map $(P \cdot \pi_{cyclical},\ P \cdot \pi_{noncyclical})$.

Proof: Direct inference gives:

$$(P,\ Q) \cdot \pi_{cyclical} = (P \cdot \pi_{cyclical},\ P \cdot \pi \cdot \pi_{cyclical}) = (P \cdot \pi_{cyclical},\ P \cdot \pi_{noncyclical}) .$$

More about permutations

Let us try to find out some more operations on permutations, but now assuming that universal set is partitioned into subsets, namely:

$$C_1 \cup C_2 \cup \ldots \cup C_k = C^{\Sigma} = C.$$

If permutation acts on C^{Σ}, we write

$$p = (C_1 : p_1, C_2 : p_2, \ldots, C_k : p_k),$$

if for all i from 1 to k p_i is injection from C_i into C^{Σ} so that images of p_i induce partitioning on C that in general is distinct from C^{Σ}.

Further we are dealing with practical case when $C^{\Sigma} = C_1 \cup C_2$.

It is convenient for us in place of universal set C consider two isomorphic nonintersecting sets C and $\overline{C}$ with bijection u from u C to $\overline{C}$.

Thus, let $C \cup \overline{C}$ be universal set so that $C \cap \overline{C} = \varnothing$, and u is bijection from C to $\overline{C}$; with $\overline{e}$ denoting $u(e)$. If permutation p acts on C [i.e., $C^{p} = C$], then we may wish sometimes to extend p on $C \cup \overline{C}$; so that this extension were $p_{ext} = (C : p, \overline{C} : p')$. We would write for this new extended permutation

$$p_{ext} = \begin{cases} C : c \mapsto c^{p} \\ \overline{C} : \overline{c} \mapsto \overline{c}^{\,p'} \end{cases} = \begin{cases} C : p \\ \overline{C} : p' \end{cases}.$$

Let us consider trivial extension with identity permutation on $\overline{C}$: $p = (C : p, \overline{C} : e)$.

Further, for permutation p we define other extension $\overline{p}$, so that for $c \in C$ and $c^{p} = d$ and $\overline{c} \in \overline{C}$, there should hold $c^{\overline{p}} = c$ [i.e., identity permutation] and $\overline{c}^{\,\overline{p}} = \overline{d}$ [i.e., isomorphically induced from C to $\overline{C}$ by u]:

$$\overline{p} = \begin{cases} C : c \mapsto c \\ \overline{C} : \overline{c} \mapsto \overline{c}^{\,\overline{p}} \end{cases}$$

It is easy to see that holds

$$\overline{p} = \begin{cases} C : e \\ \overline{C} : \overline{p} \end{cases} = \begin{cases} C : e \\ \overline{C} : u \cdot p \cdot u \end{cases} = u \cdot p \cdot u.$$

For permutation p we define twine permutation $\tilde{p}$ in order that holds

$$\tilde{p} = \begin{cases} C : c \mapsto \bar{c} \\ \bar{C} : \bar{c} \mapsto c^p \end{cases}.$$

Thus, $\tilde{p}$ should be equal to $(C : u, \bar{C} : \bar{p} \cdot u)$. It is easy to see that $\bar{p} \cdot u = u \cdot p$. From here we get

$$\tilde{p} = \begin{cases} C : u \\ \bar{C} : u \cdot p \end{cases} = u \cdot p = \bar{p} \cdot u .$$

Let us prove technical lemma that helps to deal with permutations in some specific cases:

Lemma. Let $C^{p_1} = C$ and $C^{P_2} = C$ and $u : C \to \bar{C} : c \mapsto \bar{c}$, so that $\bar{u} : \bar{C} \to C : \bar{c} \mapsto c$. The there holds

$$p = \begin{cases} C : p_1 \cdot u \\ \bar{C} : u \cdot p_2 \end{cases} = p_1 \cdot u \cdot p_2 .$$

Proof follows from direct calculation.

Calculation of image of partial map and its characteristics

Let us use new operations on permutations in order to calculate characteristics of maps.

First we find how to calculate map's image.

Permutations act on C. Corners that appear in image we attribute to set $\bar{C}$, that comes with bijection $u : C \to \bar{C} : c \mapsto \bar{c}$. Let $u' : p \mapsto \bar{p}$ be extension of bijection, so that diagram commutes

$$\begin{array}{ccc} c & \xrightarrow{u} & \bar{c} \\ \downarrow & & \downarrow \\ c^p & \xrightarrow{u} & \bar{c}^{\bar{p}} \end{array}$$

Then for permutation for which holds $C^p = C$, we write

$$p = \begin{cases} C : p \\ \bar{C} : e \end{cases}; \quad \bar{p} = \begin{cases} C : e \\ \bar{C} : up \end{cases} = upu .$$

This way defined bijection u coincide with that what we defined for image of p-map, if only for twine permutation holds:

$$\tilde{p} = \begin{cases} C : u \\ \overline{C} : \overline{p}u \end{cases} = \begin{cases} C : u \\ \overline{C} : up \end{cases} = up = \overline{p}u \, .$$

Indeed, we defined twine permutation starting from expression

$$\tilde{p} = \begin{cases} C : c \mapsto \overline{c} \\ \overline{C} : \overline{c} \mapsto c^{p} \end{cases} .$$

But, just this form of definition of twine permutation were required in order it would coincide with that what we defined by entering image of p-map. Indeed, if orbit of p is $c = (c_1 ... c_k)$, then corresponding orbit of $\overline{p}$ is $\overline{c} = (\overline{c}_1 ... \overline{c}_k)$, and corresponding orbit of $\tilde{p}$ is $\tilde{c} = (c_1 \overline{c}_1 ... c_k \overline{c}_k)$.