4-critical wheel graphs of higher order.

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Abstract

4-critical wheel graphs of higher order are considered concerning their belonging to free-planar or free-Hadwiger classes.

We are using terminology from [1, 3, 4].

In [2] Jose Antonio Martin Hernandez suggested to use the theory of free minor closed classes of graphs in trying to prove the Hadwiger conjecture. We present here some examination in this direction.

For $k > 1$, let us call Hadwiger class of order k $H_k = N_0(K_k)$, i.e. the one generated by the forbidden graph K_k. Then, following many researchers, Hadwiger conjecture may be formulated as follows: In the Hadwiger class of order $k > 1$ graphs are $k-1$-colorable. For $k < 7$ this assumption is proved to be true, but other cases remain hypothetical.

In [2] J.A.M. suggested to examine whether k-critical graphs belong to the class $Free(H_{k+1})$ [3, 4]. [Graph belonging to $Free(H_{k+1})$ we call free-Hadwiger graph.] Following Kratochvil theorem [3, 4], $Free(H_{k+1})$ would be the class without minors K_{k+1}^- and $K_{k+1}^{\odot}$, i.e. K_{k+1} without edge or with split vertex.

It is easy to see that this assumption works for cases $k < 4$. Further, we present sequence of 4-critical graphs, G_9 (fig. 1), G_7 (fig. 2), G_5 (fig. 3), G_3 (fig. 4), where none of the graphs belong to the class $Free(H_5)$, i.e. they are not free-Hadwiger graphs because they all contain the minor K_5^-. Thus, the suggestion in [2], at least for case $k = 5$, could be declined. Nevertheless, we find it useful that help us to forward some new conjectures. These newly discovered 4-critical graphs may be arranged in a sequence where each of them is a minor of the following, i.e.,

$$G_3 \prec G_5 \prec G_7 \prec G_9.$$

Indeed, it is easy to see that G_7 may be received from G_9 by contracting two successive edges on the sides. The same applies to the inclusive pairs $G_5 \prec G_7$ and $G_3 \prec G_5$.

This sequence may be replenished with wheel graphs W_i, $i = 3, 5, 7, 9$, as minors of these graphs. See what we get in fig. 5. We use here almost trivial fact that $W_i \prec G_i$ for $i = 3, 5, 7, 9$.

All graphs G_i, $i >= 3$ may be considered as first order higher wheels. All they have as minor K_5^-. Thus, they all have *minor brackets* $< K_5^-, K_5 >$ and $< K_{3,3}^-, K_{3,3} >$ where minor bracket for graph G we define as a pair of graphs $< a, b >$ where a is minor of G, but b isn't. On the other side, ordinary [zero order]wheels W_i, $i > 4$ have minor brackets $< W_4, K_5^- >$ and $< C_6^+, K_{3,3}^- >$. Besides, all these minor brackets are simple, i.e., tightest possible ones in the very natural sense. Moreover, higher than first order wheels should have the same minor brackets as first order wheels.

Further, we raise the question 1) are the only 4-critical graphs that belong to $Free(H_5)$, [zero order] wheel graphs and wheels with split edges [see fig. 6], i.e. 2) are all higher

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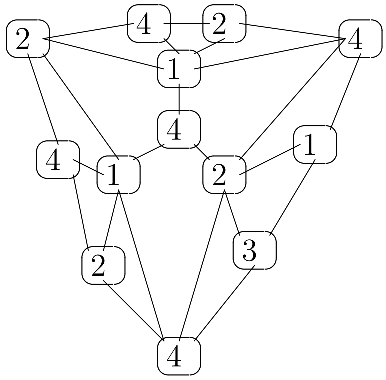

Figure 1: Example of 4-critical graph that is not free-planar, not even free-Hadwiger graph. This is a first order higher wheel G_9.

order wheels non-free Hadwiger graphs? Thus, does there exist a critical free-Hadwiger non-free-planar 4-critical graph?

Besides, we make some judgements concerning graph G_3. It is easy to see that G_3 is the 'cube with one corner cut off' graph. We could ask about higher order cube graphs, what we could get after cutting off some of cube's corners? For 4-cube, it is rather easy to see that cutting off corners [one, two, three] can't give any 5-critical graph. See, for example, one case of 4-cube graph with two corners of the cube cut off on fig. 7. Judging from the 4-cube graph experience, we conjecture that none of the higher order cube graphs with cut off corners of the cube can be a critical graph.

References

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[2] Jose Antonio Martín Hernandez, *On Hadwiger conjecture*, personal e-mail communication, 2006.

[3] Kratochvíl J. *About minor closed classes and the generalization of the notion of free-planar graphs*, personal communication, 1994, 2pp.

[4] Zeps D. *Free Minor Closed Classes and the Kuratowski Theorem*, KAM Series, 98-409, Prague, 1998, 10 pp.

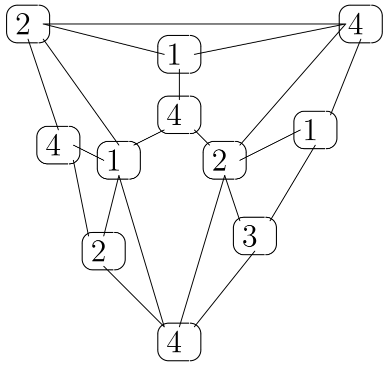

Figure 2: First order higher wheel G_7. It is obtained from G_9 by contracting two of its side edges.

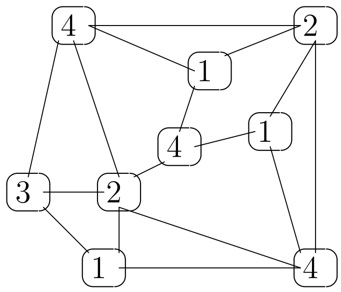

Figure 3: First order higher wheel G_5. It is obtained from G_7 by contracting two of its side edges.

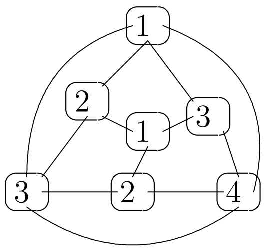

Figure 4: First order minimal possible higher wheel G_3. It is obtained from G_5 by contracting two of its side edges. It is minimal in the sequence of these graphs obtained by contractions giving graph distinct from K_4. Next graph by contractions should be K_4. G_3 has minor K_5^-, thus, it is not Free-Hadwiger graph. Besides, it is easy to see that G_3 can be imagined as the cube with one corner cut off.

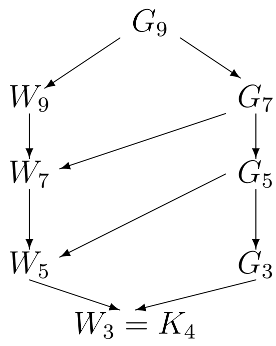

Figure 5: Lattice of 4-critical graphs, where arrows show reductions of graphs by contractions of edges. In place of G_9 may stand any G_i with odd i. Any next column in the lattice would be wheel graphs with higher order.

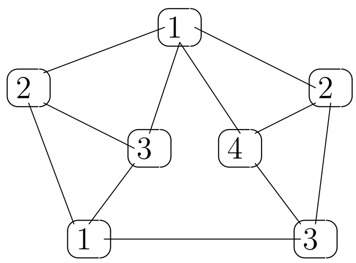

Figure 6: 4-critical graph obtained from W_5 by spitting of an edge [or spoke of the wheel]. The operation of the edge splitting preserving 4-criticality is easy to be generalized.

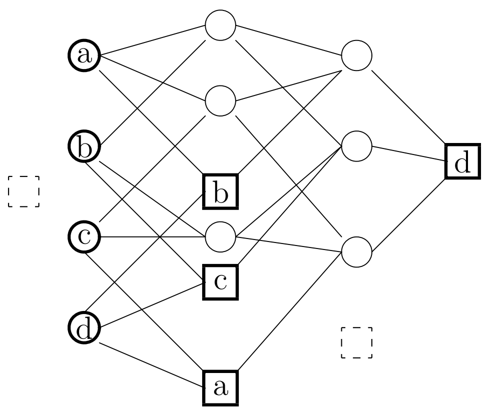

Figure 7: 4-cube with two corners [dash boxes] cut off. Marked vertices, circles and boxes correspondingly, are neighbors of cut off corners by hyperplane and thus all connected via edges [not drawn in the figure]. Letters denote distinct colors of vertices. It is easy to see that the graph is not 5-chromatic.