Combinatorial maps theory as mathematical challenge.

Dainis ZEPS *

Abstract

We argue that combinatorial maps theory is mathematical challenge that is not sufficiently developed but rather is some unreasonably unattended or weakly supported by interest of researchers area of mathematics.

1 Introduction

Since year 1993 we have been working in the area called combinatorial maps [11, 30, 31, 32, 33, 34, 35, 36, 37, 38]. In Ph.D. thesis [35] we suggested that combinatorial maps may be used as tool for building graph theoretical algorithms. Up to now it is clear that combinatorial may be used as tool to describe and code graph and some set of its functions. However, if set of these functions were developed to sufficient size it could acquire computational capacity.

2 Main idea

Computational map theory we could define as mathematical theory where the only legal tool would be rotation.

One rotation is permutation and the corresponding class symmetric group of permutations. As it is shown in [32, 35], two rotations define graph, but actually one rotation is sufficient to define graph. Thus we gain that class of permutations has in correspondence class of graphs. In general, two rotations describe graph, three rotations describe partial map that corresponds to hypergraph.

In [12] combinatorial maps are generalized to k-order combinatorial maps or constellations defined by k rotations.

2.1 About book [12]

Full information on [12]: S. K. Lando and A. K. Zvonkin, Graphs on Surfaces and Their Applications (with Appendix by Don B. Zagier), Springer-Verlag, 2004. – XVI+455 pages; ISBN 3-540-00203-0.

Authors describe their book as follows:

This book is an introduction to certain new directions in the (very classical) theory of combinatorial maps, otherwise called embedded graphs. Namely, we consider the theory of "dessins d'enfants", matrix integrals in map enumeration and in quantum gravity, moduli spaces of algebraic curves, Hurwitz spaces, and Vassiliev knot invariants. The Appendix written by Don Zagier presents a concise introduction to finite group representation theory and to its applications to enumeration problems.

*Author's address: Institute of Mathematics and Computer Science, University of Latvia, 29 Rainis blvd., Riga, Latvia. dainize@mii.lu.lv

We would like to point out about this book this. Higher order combinatorial maps are used as necessary tool to describe ramified coverings and developing this direction may considerably change the appearance of theory of holomorphic functions.

2.2 Combinatorial maps as description of algorithms

Up to now combinatorial maps are sufficient do describe and code graph or set of graphs and their simplest functions. Among the most useful are knot of the graph and another function called knotting. In [38] is shown that knot is very powerful function among rotations that describe graph and may be developed in a considerable set of novel functions for combinatorial structures.

Further, we expect according ideas expressed in [35] that sufficiently developing set of functional rotations their set could be sufficient to build algorithms. In articles [33, 36] this idea is developed using simplest rotations that were known to the moment.

Let us take some full description of some graph theoretical algorithm using, say, some programming language expressed in corresponding set of instructions and description of corresponding data structures. If we could both the set of instructions and description of data structures replace with rotational functions, we were able to replace the running of the corresponding algorithm by computation some set of rotations, say, permutations. Up to now we do not possess sufficiently amount of powerful rotational functions, or they are too simple, to rewrite graph theoretical algorithm in rotations, i.e., to use such proposed schema in practical applicable way.

We propose following way to progress. First, we may try to describe graph theoretical algorithm with set of invariants which are rotations. Thus, both actions described by computer instructions and data should be step by step replaced with rotational functional descriptions and at the end expressed as rotations i.e., combinatorial maps. Thus, the problem should be attacked from both ends. First, we should develop effective rotational functional tool kit and learn how to describe our algorithms with rotational invariants.

Author has tried to work in this direction. He took optimized algorithm of calculation of graph's genus. Step by step it is possible to develop rotational invariants both for actions and data, but the work turned to be incredibly large and at the end came natural conclusion that this work is not possible to do for one either few human beings. It requires fundamental investment of brain activity of many people.

Up to now there are few people who work on combinatorial map theory. In the references lower there are named most of people and most of their achievements in this direction, except that what concerns permutations and their groups. The list is very short. It is because of very little understanding, except maybe of people who directly work in the area, what this direction could give.

The author would like to give comparison with complex numbers. If one tried to name all authors and works in this directions it were impossible. Evidently, there should tick away sufficiently long time to understand that rotation functions [not only as permutation groups] are as fundamental as complex numbers. Let us turn attention to solitons which may be considered as simple rotations that tend to be rotations perpetually. They are used by some authors in differential equations in geometrical characterizations of theirs, but the highest usefulness of this, as it seems, is known only to these authors. Lastly, solitons are used in description of quarks in SM in theoretical physics. Thus, in nature all consists from stable indestructible rotations. But let us return to complex numbers.

From sines and cosines there may be built all, but sines and cosines are nothing other than rotations.

The distinction between simple investigation of permutations [and their groups] without structuring them and their structuring in some sets with designations is considered in the book [16]. First way is well represented by investigation of permutations in all possible aspects. Second way or rather many ways could give many new developments among one possible way would be to build environment [only from rotations] to keep together all features proved by theorems where these same theorems may be run as routines and so on.

In [38] we are going to show that only one operation, fixing of knot in the map, produces several types of rotations around the map. These types are at least five: rotations of knot type, knot square type, knot square with orbits in both directions, colored involutions and more. Each of them possess its own algebra. Types of rotation should be first step towards structuring of eventual rotational environment.

Further, looking for eventual rotational functional environment, we may compare situation with language of predicate calculus Prolog. The same what is done with predicates implementing them into PROLOG like environment, where predicates may be run as routines, this same may be done, or must be done in perspective, with rotations. But there should be great difference between Prolog like environment and rotations running environment, where Prolog gives programmer feasibility to write arbitrary predicates where Prolog running should provide corresponding predicate calculations, but in rotations running environment programmer codes all possible theorems on rotations and aggregates some subset of them to perform some desirable calculations. To attain such environment we should acquire sufficiently theoretical experience in terms of large set of rotational theorems.

Author is using computer program to test results developing the theory of combinatorial maps and uses to run experiments with random generated permutations. This program may be used as a starting point to project programming language imitating system where there only rotations work to run algorithms.

3 DARPA Mathematical Challenges

In [39] let us see DARPA Mathematical Challenges and find the sixth challenge which runs:

Mathematical Challenge Six: Computational Duality

* Duality in mathematics has been a profound tool for theoretical understanding. Can it be extended to develop principled computational techniques where duality and geometry are the basis for novel algorithms?

Thus, we raise the combinatorial map theory challenge as expressed in this formulation associating it with what was said upper.

4 Combinatorial maps theory versus traditional graph theory

There is simple way to distinguish between traditional graph theory and maps theory in its combinatorial outlook. What we do in graph theory may be easy visualized. Actually, graph theory is systematic aggregation of some our visual experience in its simplest capacity, i.e., what concerns simple relations. Graph theory thus starts with visual pictures that are easy to be captured and routines performed in these pictures are as like easy conceivable.

What we do in combinatorial maps theory? We are translating all features of graphs in terms of rotations. And there is the point where we loose visual intuitive picture that was with us looking on graph as a picture.

Thus, we may conclude that graph theory mostly allows to work with visualizable constructs whereas in combinatorial theory, where rotations come as expressions of propositions, visualizability is lost completely. It may be compared with work in the darkness.

5 Combinatorial maps theory versus permutational group theory

Combinatorial maps theory deserves to be separated from traditional permutational group theory. Best way to persuade ourselves that combinatorial maps theory is worth of this is to build computational environment where complicate graph theoretical algorithms may be run.

6 Conclusions

We conclude that combinatorial map theory may be developed in the direction where rotational functions are not only examined and proved as theorems but, appropriately structured, used as environment for calculations, but before that combinatorial maps should be investigated in the way where they are structured in sets of designated rotations thus acquiring computational feasibility to run algorithms in them.

Although there are two fundamental books devoted to this subject, i.e., [1] of P. Bonnington and Ch. Little and other [12] of S. Lando and A. Zvonkin, too little attention is given to this area which, in comparison with permutation group theory and traditional graph theory, could give incredibly great development in combinatorics. The direction deserves to be distinguished from both them.

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