The use of combinatorial maps in graph-topological computations

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Abstract

Having through the use of combinatorial maps a one-one correspondence between, say, permutations and graphs on surfaces, we try to find out simple formulas with permutations for non trivial calculations in the graphs on surfaces.

1 Introduction

We continue to investigate combinatorial maps, see [1, 3, 4, 6, 7, 8, 9], applying the same idea of considering the corners between the edges in the embedding of the graph on the surface to be the elements on which act permutations [11].

Using permutations as a tool for combinatorial map, we can get a one-one map between permutations and graphs on surfaces. One way, how to do this, is shown in [11]. A way, how to exploit this, is for some chosen feature of permutations to find the corresponding one in the graphs. Reversely, if we can find for operations on graphs corresponding operations on permutation, then we can hope, that some nontrivial manipulations on graphs can be done using the simplest operations on permutations, i.e. multiplication, selection of submaps etc. In this work we argue to have proved a simple conjecture on permutations with less trivial graph-topological conjuncture in correspondence, but the main stress putting on the possibility to do corresponding graph-topological calculus using simple operations with permutations.

We have implemented the permutation calculus in the PASCAL program, reproving our ideas on different series of maps, that are

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entered either manually or generated randomly. In the environment in this way built, we both build the algorithms to do some computations and compute many things with the simplest formulas with permutations. The investigations are intendet to find sufficiently large set of permutational furmulas with graph-topological operations in correspondence, to use it as an independent vehicle for topological calculus already without algorithmical support. A point that must be mentioned is that permutational calculus with simplest operations can be done linear against the order of permutations or the amount of the updated part in them.

2 Permutations, combinatorial maps and partial combinatorial maps

Permutations act on a universal set C the elements of which we call corners because of their geometrical interpretation. For a permutation P and $c \in C$ c^P denotes the element of C in which P maps c. Permutations we multiply from left to right.

In general we use the same terminology for combinatorial maps as in [11].

A pair of permutations (P, Q) we call *combinatorial map* whenever $P^{-1} \cdot Q$ is a matching, i.e. involution without fixed elements. The main characteristics of combinatorial map is its edge-rotation ϱ equal to $Q \cdot P^{-1}$ and next-edge-rotation π equal to $P^{-1} \cdot Q$.

Usually we are working with classes of maps with fixed π, that are closed under multiplication of maps from left.

Partial combinatorial maps (shorter *p-maps*) are pairs of permutation without any restriction on their multiplication, i.e. their edge rotation can be arbitrary permutation.

3 Paths and cycles in combinatorial maps

For a combinatorial map (P, Q), a sequence of elements $c_1, \ldots, c_n$ $(n > 0)$ is *a path*, if for c_i, $(0 < i < n)$, next element c_{i+1} in the sequence is equal to c_i^P or c_i^Q. If this same holds also for c_n and c_1, i.e. c_1 equals to c_n^P or c_n^Q, then the path is closed, forming *a* *cycle*. *Simple* cycles are cycles without repetition of elements. For a combinatorial map simple cycles are transitive permutations on some subsets of the set C with the appropriate choice of next elements.

A cycle cover of the combinatorial map (P,Q) is a permutation acting on the whole set C, where each of its orbits is a cycle in (P,Q). Then cycles in cycle covers are always simple cycles. Simplest cycle covers of (P,Q) are P and Q themselves.

Trivially enough, multiplying a cycle cover of some combinatorial map with some submap [11]of its π, i.e. next-edge-rotation, we get another (and possibly every) cycle cover of this combinatorial map.

We say, that two cycles ζ_1 and ζ_2 in (P,Q) *touch* each other when for same element e_1 of ζ_1 e_1^P or e_1^Q or e_1^{-P} or e_1^{-Q} is passed through by the cycle ζ_2. Let us suppose, that the elements of C for some combinatorial map (P,Q) with a fixed cycle cover are colored in such a way, that 1)elements of cycles of the cycle cover are colored in the same color, and 2) cycles with equal element coloring do not touch. Such a coloring of elements of combinatorial map we call *a cycle cover coloring.*

4 Two-colorable cycle covers in combinatorial maps

Firstly let us consider two-colorable cycle covers generally.

Theorem 1 *Let us suppose, that for a combinatorial map (P,Q) with cycle cover ζ elements are colored in two colors, so that this is also a cycle cover coloring. Let an arbitrary edge be with its (possibly not all distinct) corners c_1, c_2, c_3, c_4, such that $c_2 = c_1^{-P}$, $c_3 = c_1^{\pi}$ and $c_4 = c_1^{-Q}$.*
Then there are three possibilities:
1) c_1 and c_2 belong to the same color and c_3 and c_4 to the other color , [then we call such an edge **cut-edge***];*
2) c_1 and c_4 belong to the same color and c_2 and c_3 to the other color, [then we call such an edge **cycle-edge***];*
3) all corners of the edge are of the same color,[then we call such an edge **inner edge***].*

Proof An edge is either a place where cycles of different color touch, or the same cycle meets itself (going either along the edge [like cycle-edge] or across the edge [like cut-edge]).

Let us notice, that , when the edge is not an inner edge, the pairs (c_1, c_3) and (c_2, c_4) contain corners of different coloring. So, edges of one color in π and ϱ are inner edges, but with different color corners - cycle-edges and cut-edges.

For further considerations most useful are cycle cover colorings without inner edges. Then immediately is right what follows.

Theorem 2 *Let us suppose, that for a combinatorial map (P, Q) with cycle cover ζ elements are colored in two colors, green and red, so that this is also a cycle cover coloring and there are not inner edges.*
Then $\mid C_{green} \mid = \mid C_{red} \mid$, where $C_{green} \cup C_{red} = C$, and π and ϱ are one-one matches between C_{green} and C_{red}.

Further we speak about the way how to get two-colorable cycle covers without inner edges in combinatorial maps, using a knot of this map. Combinatorially, knot is a zigzag walk cover in the combinatorial map [1] and [11]. Zigzag walk always has orbits of even degree, so it is possible to connect with a zigzag walk a coloring of elements of combinatorial map. More over, elements of one color form cycles of one color, resulting in the cycle cover with coinciding coloring.

Theorem 3 *Let for a combinatorial map (P, Q) is given coloring $C_{green} \cup C_{red}$, and a knot of this map by this same coloring is colored alternatively. Then there exists one unique cycle cover of this map with this same coloring as the cycle cover coloring without inner edges.*

Proof Let us suppose the opposite, and elements of one color do not form cycles. Then there must be an element c, say of green color, but the colors of both $c^P (= c_1)$ and $c^Q (= c_3)$ are red. But the pair (c_1, c_3) belongs to π and its elements must have different colors. We have come to a contradiction. And of course, there can not be inner edges.

Thenafter, inversely, each cycle cover without inner edges fixes some knot with precision to the reverse.

Let for a chosen knot μ of (P,Q) the corresponding cycle cover is ζ and let us express it as $\zeta_{green} \cdot \zeta_{red}$, where ζ_{green} contains cycles of green elements, and ζ_{red} contains cycles of red elements.

It is easy to get following features of cycle cover ζ without inner edges for (P,Q).

Theorem 4 *It holds*
1) $\zeta \cdot \pi = \zeta_{altern}$*, where* ζ_{altern} *is a cycle cover with alternating coloring of its elements;*
2) $\zeta^{-1} \cdot P = \zeta_{altern}^{-1} \cdot Q = \pi_{cycle}$*, where* π_{cycle} *have all cycle-edges and only them;*
3) $\zeta_{altern}^{-1} \cdot P = \zeta^{-1} \cdot Q = \pi_{cut}$*, where* π_{cut} *have all cut-edges and only them;*
4) π_{cycle} *and* π_{cut} *are complementary involutions and thus* $\pi_{cycle} \cdot \pi_{cut} = \pi$*.*

5 Graphs on surfaces

Now we may think in terms of graphs and topology, but corresponding manipulations do in maps, i.e. permutations.

Let us partition C into $C_{cycle} \cup C_{cut}$, where the first set contains elements of π_{cycle} and the second - of π_{cut}.

Theorem 5 *The partial map* $(P,\zeta) \mid_{C_{cycle}}$ *is a combinatorial map on* C_{cycle}*.*

Applying the theorem dually, we get, that also (Q,ζ) restricted on C_{cut} is a combinatorial map on this set.

Let us write ζ_{cycle} in the place of $\zeta \mid_{cycle}$, and so also ζ_{cut} in the place of $\zeta \mid_{cut}$.

We call $(P,\zeta) \mid_{C_{cycle}}$ and $(P,\zeta) \mid_{C_{cut}}$ correspondingly a *cycle graph* and a *cut graph*. The second one is in general a partial map.

Before we come to our main theorem we must notice what follows.

Theorem 6 *All cycles of the cycle cover for* (P,Q) *are also cycles in p-map* (ζ,Q)*.*

Theorem 7 *(Main theorem)*

$$\gamma_{(P,Q)} > \gamma_{(P,\zeta)} + \gamma_{(Q,\zeta)}.$$

Proof Direct use of Eiler formula gives that there must hold an inequality

$$\| \zeta \cdot P^{-1} \| + \| \zeta \cdot Q^{-1} \| + 2 \| \zeta \| > 3/2 \cdot l + 2c_{(P,\zeta)} + 2c_{(Q,\zeta)} - 2,$$

i.e. $\| \zeta \| > c_{(P,\zeta)} + c_{(Q,\zeta)} - 1$; where $c_{(P,\zeta)}$ and $c_{(Q,\zeta)}$ are numbers of components in the corresponding partial maps. But cycles of the cycle cover of (P,Q) are also orbites of both (ζ, P) and (ζ, Q), but being cut as a separate component only in the one of them. This proves what was stated.

Further we try to show that this rather simple fact about permutations causes less trivial consequence in graphic-topological view.

5.1 Cutting surface along cycles in graph embeddings

From the graph topological point of view, when we cut a surface, in which a graph is embedded, along some cycle, then the surface is either cut to two parts or its genus is reduced depending whether the cut line along the cycle is contractible in the topological sense to point or not. Choosing some cycle in cycle cover and cutting orbits into orbits of one color in P along this cycle do the same thing in the combinatorial maps.

Let us denote multiplication $P \cdot P_{cycle}^{-1}$ by P and consider it nearer, seeing behind the partial map (P,Q) one with cut embendance surface along the cycles of cycle cover.

Theorem 8 *P, i.e. $P \cdot P_{cycle}^{-1}$ is equal to $\zeta \cdot \zeta_{cycle}^{-1}$ and the genus of (P,Q) is equal to the genus of (ζ, Q).*

Proof First we prove some lemmas.

Lemma 1 *Orbits of P_{cycle} have an alternating coloring, but orbits of P have elements of one color.*

Lemma 2 $P_{cycle} \cdot \pi_{cycle} = \zeta_{cycle}$.

Proof Before it is shown that $P \cdot \pi_{cycl} = \zeta$. Restricting this expression onC_{cycle} we get $(P \cdot \pi_{cycl}) |_{cycl} = \zeta_{cycl}$, but the left side is equal to $(P_{cycl} \cdot \pi_{cycl}$, what was to be proved.

Let us prove that $\zeta \cdot \zeta_{cycl}^{-1} = P$. Really,

$$\zeta \cdot \zeta_{cycl}^{-1} = P \cdot \pi_{cycl} \cdot \pi_{cycl} \cdot P_{cycl}^{-1} = P \cdot P_{cycl}^{-1} = P.$$

It remains to prove that $\gamma_{(P,Q)} = \gamma_{(\zeta,Q)}$. It follows from the fact, that multiplication of ζ by reverse ζ_{cycle} do not changes the genus of the partial map (ζ, Q), because the orbits of the reverse of ζ_{cycle} are subedges of multiedges of p-map (P, Q), where the last are the remnants of the cutting the surface along the cycles.

This completes the proof of the theorem.

6 Conclusions

Let us compare our main theorem and the topological theorem.

The main theorem above says that the difference between genera of (P, Q) and (ζ, Q) plus the difference between genera of (Q, P) and (ζ, P) is greater then genus of (P, Q).

Trying to translate this fact in the topological language, it would mean, that every cover of cycles ζ have enough cycles in the sense that, cutting along them surface both of (P, Q) and its dual (Q, P), reduces its genus completely. The uncontractable to point cycles of ζ_{cycle} in the the graph (ζ, Q) give uncontractable cycles, but in the graph (ζ, P) genus-reducing-cuts of edges.

This gives a hope, that it is possible to find more useful operations, which would result in some permutational calculus with topological application.

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References

[1] Bonnington, P., Little, C.H.C. *Fundamentals of topological graph theory,* Springer-Verlag, N.Y.,1995.

[2] Edmonds, J.K. A combinatorial representation for polyhedral surfaces, *Notices Amer. Math.Soc.,*7(1960), 646

[3] Little, C.H.C. Cubic combinatorial maps, *J.Combin.Theory Ser.B* 44 (1988), 44-63.

[4] Liu, Yanpei A. Polyhedral Theory on Graphs. *Acta Mathematica Sinica, New Series,*1994, Vol.10,No.2,pp.136-142. 275-298.

[5] Stahl, S. Permutation-partition pairs: A combinatorial generalisation of graph embedding, *Trans Amer. Math. Soc.,* (259) (1980), 129-145.

[6] Stahl, S. A combinatorial analog of the Jordan curve theorem,*J.Combin.Theory Ser.B* 35 (1983), 28-38.

[7] Stahl, S. A duality for permutations,*Discrete Math.* 71 (1988), 257-271.

[8] Tutte, W.T. Combinatorial maps, *Graph theory,* Addison-Wesley, Menlo Park, Calif., 1984.

[9] Vince, A. Combinatorial maps, *J.Combin.Theory Ser.B,* 34 (1983), 1-21.

[10] White, A.T. Graphs, *Groups and Surfaces.* North-Holland,P.C. 1973.

[11] Zeps, D. Graphs as rotations, *KAM Series,96-327* Prague,1996.

[12] Zeps, D. Graphs with rotations:partial maps, *submitted in KAM Series,*1996.