# TRYING TO PROVE THE KURATOWSKI THEOREM FROM BELOW.

DAINIS ZEPS

Abstract. This note examines possibility to prove the Kuratowski theorem from below, i.e. assuming that Kuratowski-like theorem for free-planar graphs is right. Version of Kuratowski theorem for 3-connected components is proved. Another proof of Kuratowski teorem is added.

Graph is defined as a pair of sets $(V, E)$, where $V$ is the set of vertices and $E$ – the set of edges. For graph $G$ V(G) is its vertex set and $E(G)$ is its edge set. We denote by $G-e$ graph obtained by deleting edge $e \in E(G)$ from $G$. Similarly, $G-v$ is graph obtained by deleting vertex $v \in V(G)$ from $G$.

Similarly, $G.e$ is graph obtained by contracting edge $e \in E(G)$ in $G$. Reverse operation to edge adding and its contraction is the *vertex split* operation $G \odot u$, that is not unique. Thus, if in $G$ by adding and contracting $e \notin E(G)$ appears a new vertex $u \in V(G')$ then there exists such vertex split $G' \odot u$ that we receive back previous graph $G$.

$H$ is *subgraph* of $G$ (denoting it $H \subset G$) if there is such a graph $H'$ isomorphic to $H$ and $V(H') \subset V(G)$ and $E(H') \subset E(G)$.

$H$ is a *minor* of $G$ (denoting it $H \prec G$) if $H$ can be obtained by edge contractions from some subgraph of $G$. It is easy to see that if $H \prec G$ then $H$ can be obtained from $G$ by vertex deletions, edge deletions and edge contractions.

A class of graphs $A$ is called *minor closed* if for each graph $H$ belonging to $A$ and arbitrary graph $G$ from $G \prec H$ follows that $G$ is in $A$.

For a minor closed class $A$, $F(A)$ is the minimal set of forbidden minors, i.e.

$$F(A) = \lfloor \{G \mid G \notin A\} \rfloor.$$

Here we use a notion $\lfloor B \rfloor$ denoting set which contains only minimal minors of $B$:

$$\lfloor B \rfloor \triangleq \{G \mid H \in B \wedge H \prec G \Rightarrow H \cong G\}.$$

---

*Date*: 28. october 2000.

THIS RESEARCH IS SUPPORTED BY THE GRANT 97.0247 OF LATVIAN COUNCIL OF SCIENCE.

Analogously, $\lceil B \rceil$ which contains only maximal graphs, i.e. all graphs of $B$ that are minors of $\lceil B \rceil$ is defined as follows:

$$\lceil B \rceil \triangleq \{G \mid H \in B \wedge G \prec H \Rightarrow H \cong G\}.$$

**Proposition 0.1.** *For a minor closed class $A$ if $G$ doesn't belong to $A$ there exists such $H \in F(A)$ that $H \prec G$ and conversely.*

**Theorem 0.2.** *(Robertson, Seymour): $F(A)$ is finite for any minor closed $A$.*

Let $\mathcal{G}(n, m)$ denote the set of all graphs with $n, (n > 0)$ vertices and $m, (m \geq 0)$ edges. Let $\mathcal{B}(n, m)$ denote the set of all bigraphs with $n, (n > 0)$ vertices and $m, (m \geq 0)$ edges. Let $\mathcal{C}(n, m)$ be arbitrary subset of $\mathcal{G}(n, m)$ and $\mathcal{D}(n, m)$ be arbitrary subset of $\mathcal{B}(n, m)$. Sometimes we are saying that graph $G$ with $n$ vertices and $m$ edges belongs to a set of graphs (not specifying the set) meaning by this set $\mathcal{G}(n, m)$.

Sets of graphs $\mathcal{C}_1 = \mathcal{C}(n_1, m_1)$ and $\mathcal{C}_2 = \mathcal{C}(n_2, m_2)$ are *non-compatible* if no graph from $\mathcal{C}_1$ is a minor for any graph from $\mathcal{C}_2$ and vice versa.

**Lemma 0.3.** *Sets of graphs $\mathcal{G}_1 = \mathcal{G}(n_1, m_1)$ and $\mathcal{G}_2 = \mathcal{G}(n_2, m_2)$ are non-compatible iff $n_1 > n_2$ and $m_1 < m_2$ or vice versa: $n_1 < n_2$ and $m_1 > m_2$.*

*Proof.* If there holds $n_1 > n_2$ and $m_1 < m_2$, then it is easy to see that compatibility is impossible.

Let us assume that $\mathcal{G}_1 = \mathcal{G}(n_1, m_1)$ and $\mathcal{G}_2 = \mathcal{G}(n_2, m_2)$ are non-compatible. If graphs with equal number of vertices were allowed then we could take a graph and its subgraph with the same number of vertices and the sets were compatible. If graphs with equal number of edges were allowed then we could take two equal graphs and one replenish with as many as necessary isolated vertices and they were compatible. Thus, the non-compatibility condition must be just as stated by lemma. $\square$

It is easy to see that the statement of the lemma is right also for pairs of sets $\mathcal{B}(n_1, m_1)$ and $\mathcal{B}(n_2, m_2)$ and $\mathcal{B}(n_1, m_1)$ and $\mathcal{G}(n_2, m_2)$ respectively. Evidently, the last case corresponds to Kuratowski graphs, i.e. $\mathcal{B}(6, 9)$ and $\mathcal{G}(5, 10)$ are non-compatible sets of graphs, where $\mathcal{B}(6, 9)$ comprise only $K_{3,3}$ but $\mathcal{G}(5, 10)$ only $K_5$.

For sets of graphs $\mathcal{G}$ and $\mathcal{S}$ let $\mathcal{G} \mid_{\mathcal{S}}$ denote only these graphs from $\mathcal{G}$ that do not have graphs from $\mathcal{S}$ as minors:

$$\mathcal{G} \mid_{\mathcal{S}} \triangleq \{G \mid G \in \mathcal{G} \wedge \forall H \in \mathcal{S} : H \not\prec G\}.$$

$N_\circ(B)$ denotes the minor closed class with $B$ as its set of forbidden minors, i.e.

$$N_\circ(B) \triangleq \{G \mid \forall H \in B : H \not\prec G\}.$$

In other words, we may say, that $N_\circ(B)$ is a minor closed class generated by its forbidden minors in $B$. For example, $N_\circ(K_5, K_{3,3})$ is the class of planar graphs, as it is asserted by Kuratowski theorem.

One more denotation for the minor-closure of $B$:

$$\langle B\rangle \triangleq \{G \mid \exists H \in B : G \prec H\}.$$

If $B = \{G\}$ then we write $\langle G\rangle$ in place of $\langle\{G\}\rangle$.

Let us use some denotations for some small graphs:

$Z_k$, $k > 0$, is empty graph with $k$ vertices, equal with $Z_k = k \times K_1$.

$X = K_{1,4} + K_1$, i.e. star with four edges and an isolated vertex.

$Y = K_{1,3} + K_{1,1}$, i.e. star with three edges and an isolated edge.

$V = K_3 + 3 \times K_1$, i.e. triangle and three isolated vertices.

$U$ is the cycle of length 5 plus a vertex of degree 2 connected to two non-consecutive vertices of the cycle.

Let us state some simple facts with these and Kuratowski graphs:

**Lemma 0.4.** $\lceil N_\circ(\{V, Z_7\})\rceil = \{K_{3,3}, K_5, K_{2,4}, K_{1,5}, U\}$.

*Proof.* If graph has 6 vertices it is without triangles, i.e. it is a bigraph. There are four possible maximal such graphs, i.e. $K_{3,3}, K_{2,4}, K_{1,5}$ and $U$.

If graph has less than 6 vertices, it is arbitrary otherwise, i.e. maximal graph is $K_5$. □

**Lemma 0.5.** $\lceil N_\circ(\{V, X, Y, Z_7\})\rceil = \{K_{3,3}, K_5\}$.

*Proof.* $X$ excludes $K_{1,5}$ and $Y$ excludes $U$ and both $X$ and $Y$ exclude $K_{2,4}$. □

If $Z_7$ is removed from lemmas condition then graphs with components isomorphic to Kuratowski graphs and their subgraphs are allowed.

A planar graph is called free-planar, if after adding an arbitrary edge it remains to be planar. In [4] it is proved, that the class of free-planar graphs is equal to $N_\circ(K_5^-, K_{3,3}^-)$, and its characterization in terms of the permitted 3-connected components is given.

In [2] a generalization of the notion of free-planar graphs is suggested. We denote by $Free(A)$ the class of graphs that consists of all graphs which should belong to $A$ after adding an arbitrary edge to them. It is easy to see, that, if $A$ is minor closed, then $Free(A)$ is minor closed too [2]. Because of this we use to say, that $Free(A)$ is *free-minor-closed-class* for a minor closed class $A$.

In [2] Kratochvíl proved the theorem:

$$F(Free(A)) = \lfloor F(A)^- \cup F(A)^\odot\rfloor,$$

where

$$B^{-} \triangleq \{G - e \mid G \in B, e \in E(G)\}$$

and

$$B^{\odot} \triangleq \{H \mid H \cong G \odot v, G \in B, v \in V(G)\}.$$

Further, we denote by $Free^k(A)$ repeatedly applied $Free$ $k$ times, i.e.

$Free^0(A) = A;$

$Free^k(A) = Free(Free^{k-1}(A)).$

Let for a minor closed class $A$ $Free^m(A)$ is not consisting of only empty graphs but $Free^{m+1}(A)$ is, then we say that $A$ is *of depth $m$*.

In the graph $G$ with some vertex $v$ a vertex split $G \odot v$ is called *proper* if both new vertices arising in the result of the split of $v$ are of degree at least two. Otherwise the vertex split is called *non-proper*.

**Theorem 0.6.** *Let a class $A$ be of depth $m$ and all graphs of $F(A)$ belong either to mutually non-compatible or coinciding sets of graphs. If there holds*

$$F(Free(A)) = F(A)^{-}$$

*then there holds also*

$$F(Free^k(A)) = F(Free^{k-1}(A))^{-}$$

*for $k = 1, ..., m$.*

*Proof.* By induction if $m$ is equal to 1 all is done by theorem's assumption otherwise it suffices to prove that

$$F(Free^{k+1}(A)) = F(Free^k(A))^{-}$$

for $1 \leq k < m$ assuming that

$$F(Free^k(A)) = F(Free^{k-1}(A))^{-}$$

is right.

Let us suppose first that $F(Free^k(A))$ consists only from one graph. Then all graphs $F(Free^k(A))^{-}$ have the same number of edges and they can't be proper minors of each other. Thus, they all are present in $F(Free^{k+1}(A))$.

$F(Free^k(A))^{\odot}$ can not give some contribution to $F(Free^{k+1}(A))$ either. Let us suppose for a moment that it does and some graph $G \in F(Free^k(A))$ is such that $G'$ with some vertex $v$ split ($G' = G \odot v$ giving new vertices $v_1$ and $v_2$) were not present in $F(Free^k(A))^{-}$.

Let us suppose that this vertex split was non-proper. Then a corresponding hanging edge or isolated edge arises, but graph without this edge is already present in $F(Free^k(A))^{-}$. Thus, non-proper vertex split can not give any new contributor to $F(Free^{k+1}(A))$. Further, let us suppose that this vertex split was proper. Let us find some ancestor $H$ of $G$ in $F(Free^{k-1}(A)$ such that $H$ minus some edge $e$ is equal to $G$. Let us add $e = (s, t)$ to $G'$ getting a new graph $H'(= G' + e')$ in this way: if none of ends of $e$ were equal to $v$ then adding $e(= e')$ is possible only in one way; if one of the ends, say $s$, of $e$ is equal to $v$ then without loss of generality we add a new edge $e' = (v_1, t)$ to one of the new vertices (i.e. $v_1$) arising in the result of the split of $v$. Then this graph must be also contributor to $F(Free^k(A))$, because there exists some vertex split $H \odot v$ such that $H \odot v$ is equal to $H'$ but the assumption of the theorem excludes this [...because $H'$ has a proper minor $h$ already present in $F(Free^k(A))^-$. If $e' \in E(h)$ then $h - e'$ should be also a minor of $G'$, thus $G'$ is not a contributor in $F(Free^{k+1}(A))$. If $e' \notin E(h)$ then $h$ minus arbitrary edge is minor of $G'$ and present in $F(Free^k(A))^-$. Thus, in this case too $G'$ is not a contributor in $F(Free^{k+1}(A))$].

Let us suppose that$F(Free^{k-1}(A)$ consists from many graphs. But, because of either non-compatibility or coincidence of the sets to which these graphs belong the same is true for the set $F(Free^k(A))$ too. Truly, non-compatible assumed contributors of $F(Free^k(A))^-$ can not exclude each other. Either can not such that are with equal number of vertices and edges except the cases of isomorphism. Thus distinct descendants of the same level from distinct forbidden graphs can not be proper minors of each other.

The consideration about the contribution of $F(Free^k(A))^{\odot}$ does not change in the general case. □

**Theorem 0.7.** *For $A = Planar$ and $0 < k \leq 10$*

$$F(Free^k(Planar)) = \mathcal{B}(6, 9-k) \mid_{\{X,Y\}} \cup \mathcal{G}(5, 10-k).$$

*Proof.* Graphs $X, Y$ exclude these minors that are present in $K_{2,4}$ but are absent in $K_{3,3}$. Indeed, $\mathcal{B}(6, 8) = \{K_{3,3}^-, K_{2,4}\}$ and $X, Y$ as minors are present in $K_{2,4}$ and its descendants and are absent in $K_{3,3}$ and its descendants. □

**Corollary 0.8.** *For $A = Planar$ Kratochvíl's theorem has the following appearance*

$$F(Free^k(Planar)) = F(Free^{k-1}(Planar))^-$$

*for $k = 1, ..., 10$.*

Let for a class $A$ of depth $m$ holds:

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$. Then we call this class $\mathcal{M}$-class.

Then we can state:

**Corollary 0.9.** *For the case $A$ is $M$-class of depth $m$ Kratochvíl's theorem has following appearance*

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

*for $k = 1, ..., m$.*

**Theorem 0.10.** *Let $B = free(A)$ and $B$ is $\mathcal{M}$-class. $A$ is $\mathcal{M}$-class iff $F(B) = F(A)^-$.*

*Proof.* If $A$ is $\mathcal{M}$-class then theorem by definiton of $\mathcal{M}$-class $F(B) = F(A)^-$. Conversely, from the definition of the $\mathcal{M}$-class and from facts that $F(B) = F(A)^-$ and $B$ is $\mathcal{M}$-class follows that $A$ is $\mathcal{M}$-class too. □

We would like to apply last theorem in the case when Kuratowski-like theorem for free planar graphs were supposed to be right and it would be necessary to prove Kuratowski theorem itself. But we could do this only in the case we knew that the class $Planar$ is $M-$class. Either we could prove Kuratowski theorem from below directly and then conclude that the class $Planar$ is $M-$class.

Let us do the latter case.

We would base ourselves on the characteristic of free planar graphs by their 3-connected components [3, 4].

Let us recall some definitions from 3-connectivity and 3-connected components [or, shorter, 3-components] [5, 3]. *Generalized* vertex or *g-vertex* is vertex or two vertices [virtual edge]. *Generalized* edge or *g-edge* is pair generalized vertices. We say that some planar 3-component becomes nonplanar *by closing g-edge* if by merging this 3-component with another proper 3-component [i. e. having $K_4$ as minor] through this g-edge the resulting graph [or 3-component] is nonplanar. See fig. 2. We say that $H$ is $S-$marked minor of $G$ if $H \prec G$ and $S$ is subset of vertices both of $H$ and $G$.

We would hold ourselves to following assumptions. Let $G$ be planar but $G$ plus some edge $e = vw$ becomes nonplanar. Let $G$ be divided into 3-components and $P_{vw}$ be minimal sequence of g-edges $v_1w_1, ..., v_kw_k$, $k > 0$ such that each g-edge belongs to distinct 3-component and $v_1 = v$ and $w_k = w$ and $w_i = v_{i+1}$ for $1 < i < k$. Let 3-component $C_i$ possesses g-edge $v_iw_i$. We say that this 3-component gives rise to nonplanarity if $C_i$ by closure of g-edge $v_iw_i$ gives graph [or 3-component] that is nonplanar. In general, by fixing such path $P$ of g-edges we would say that each component $C_i$, $1 < i \leq k$ is marked with the g-edge $v_iw_i$.

Let us consider possible components of free planar graphs. We would turn our interested to these components with such marked g-edge that by composing the graphs could give rise to nonplanarity. Let us recall

from [4] that these components with marked g-edge would be
1) $W_3$ with marked one rim edge and one spike edge in such a way that they are not touching each other;
2) $W_k$, $k > 3$, with marked two spike edges not going into common triangle;
3) $\overline{C}_6$ with marked two edges both going into different triangles.
See fig. 1 [cases 1, 2, 3].

Let us suppose the following reduction of the nonplanar graph $G$ such that $G - e$ for some $e \in E(G)$ is planar: let $G - e = G_0, G_1, ...G_i, ...G_m$ for $m \geq 0$ be such sequence of graphs that $G_{i+1} = G_i - e_i$ where not virtual edge $e_i \in E(G_i)$ and $e_i$ belongs to such 3-component of $G_i$ that is not free planar or isomorphic to $K_5^-$ and if it were giving rise to nonplanarity it remained such after elimination of $e_i$ too. Furthermore, $m$ is maximal. Let us call this reduction of the nonplanar graph $G$ with the nonplanar edge $e$ FP-reduction giving in the result graph $G_m$.

Let us formulate some variation of Kuratowski-like theorem going out from the free planarity:

**Theorem 0.11** (Kuratowski theorem from below). *Let $G$ be planar and $G + e$ nonplanar. Then arbitrary FP-reduction of $G + e$ gives $G_m$ that $G_m + e$ is nonplanar, all 3-components in $G_m$ are free planar graphs or reduced Kuratowski graphs, and at least one 3-component is, according one of four cases of fig. 1, free planar graph with pair of virtual edges giving rise to nonplanarity [cases 1,2,3], or reduced Kuratowski graph with pair of virtual vertices giving rise to nonplanarity [case 4]. In cases 1, 2, 3 $G + e$ should have $K_{3,3}$ as minor and in case 4 $G + e$ should have $K_5$ as minor. Other cases are excluded.*

Let us first prove a lemma that may be considered as sort of some variation of the Kuratowski theorem applied for 3-components.

**Lemma 0.12** (Kuratowski theorem for 3-components). *Let $C$ be 3-component with one*
*g-edge $e = vw$ and $C$ be planar but closed by $e$ becomes nonplanar and let $C$ be minimal in sense that no edge can be eliminated that $C$ reduces to some smaller 3-component with the same features. Then:*
*1) if $deg(e) = 4$ in $C$ then $C$ is isomorphic to $W_3$ [case 1 fig. 1];*
*2) if $deg(e) = 3$ and both g-vertices in $e$ are of degree 2 in $C$ then $C$ is isomorphic to $W_4$ [case 2 fig. 1] ;*
*3) if $deg(e) = 3$ and one g-vertex in $e$ is of degree 1 in $C$ then such minimal $C$ does not exist ;*
*4) if $deg(e) = 2$ in $C$ $v$ then $C$ is isomorphic to $K_5^-$ [case 4 fig. 1]*

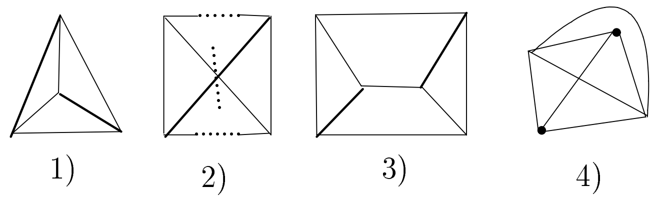


FIGURE 1. Cases of 3-components of $G_m$ in the FP-reduction giving rise to nonplanarity. Marked g-edge $vw$ is drawn bold.

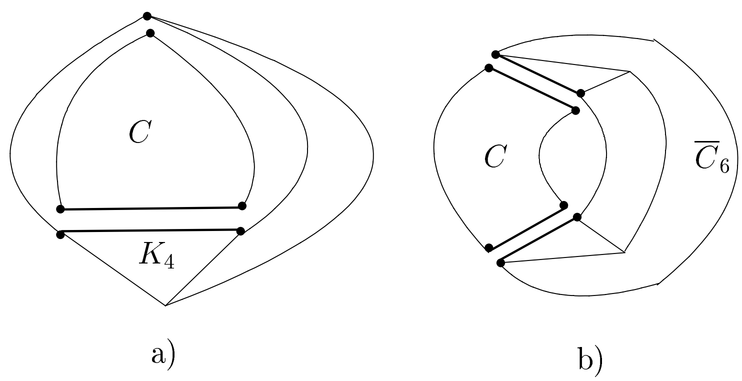


FIGURE 2. Closure of g-edge with 3-component. In case a g-edge of degree three in $C$ is closed with $K_4$; in case b g-edge of degree four in $C$ is closed with $\overline{C}_6$.

*Proof.* Because of 3-connectivity of $C$ there must be in $C$ always $\{v\}-$marked minor $K_4$ [, say, $K_v$] and $\{w\}-$marked minor $K_4$ [, say, $K_w$] as well as $\{v, w\}-$marked minor $K_4$ [, say, $K_{vw}$].

Let us consider four cases:

1) $deg(e) = 4$: $K_v$ may coincide with $K_w$ giving $W_3 \cong K_{vw}$ otherwise $C$ must have $K_4$ as $\{v, w\}-$marked minor and at least one edge not belonging to it and $C$ would be reducible not violating nonplanarity condition;

2) $deg(e) = 4$ and both g-vertices in $e$ are of degree 2: if $K_v = K_w$ then minimal possible $C$ must be isomorphic to $W_4$; otherwise $C$ must have $W_4$ as $\{v, w\}-$marked minor and at least one edge not belonging to it and $C$ would be reducible not violating nonplanarity condition;

3) $deg(e) = 3$ and one g-vertex in $e$ is of degree 1: without loss of

generality let us suppose that $deg(v) = 1$ and $deg(w) = 2$ [and $w = st$] ; then $K_v$ and $K_w$ possibly are different and $C$ in minimal case is isomorphic to $W_4$ where, say, $e_1 = vs \in E(C)$; then $C$ is not minimal because $e_1$ can be eliminated giving case 1; otherwise analogously as in case 2 $C$ would be reducible, but in this case to case 1;
4) $deg(e) = 2$: then $K_v$ and $K_w$ have 3 common edges in minimal case of $C$ being isomorphic to $K_5^-$; otherwise $C$ must have $K_5^-$ as $\{v, w\}-$marked minor plus extra edge [because of three distinct chains from $v$ to $w$ such that $v$ is not seen from $w$] and $C$ would be reducible not violating nonplanarity condition. □

*Proof of the Kuratowski theorem from below.* Let $e$ be $vw$. Firstly, in $G_m$ there exist a chain from $v$ to $w$, otherwise, if there would be an edge whose elimination would exclude such a chain, then it would comprise a 3-component and its elimination were prohibited.

We must prove that the 3-components of $G_m$ that are giving rise to nonplanarity are just as in fig. 1. Indeed, if we requested free planarity or equality to $K_5^-$ then we must get just these cases that are shown in fig. 1. Requesting additionally minimality [of edges], case 1 can not be reduced and coincides with case 1 of the lemma. Case 2 can be reduced to case 2 of lemma. In case 3 one outer edge can be eliminated giving reduction to case 1. Case 4 coincides with the case 4 of the lemma. FP-reduction was performed in such a way that the edges that were m for nonplanarity were left untouched. Thus both $G + e$ and $G_m + e$ are nonplanar because of these 3-components of $G_m$ that give rise to nonplanarity. Thus $G+e$ should have $K_{3,3}$ as minor in cases 1, 2, 3 and $K_5$ as minor in case 4. Other cases are excluded and thus Kuratowski theorem is proved. □

After we have done this excursion with 3-components of free planar graphs in order to prove the Kuratowski theorem from below we may conclude what follows.

**Theorem 0.13.** *The class Planar is $\mathcal{M}$-class.*

This follows from theorem 0.10.

What really does mean the fact that class *Planar* is $\mathcal{M}$-class? Let us accomplish following consideration. Let $G$ be planar and $G + e$ – nonplanar and let $G$ be divided into 3-components. Then according previous $G + e$ is nonplanar because of nonplanarity of some 3-components [in case of appropriate closing by g-edge]. Let the number of such 3-components be $k$, $k \geq 1$. Then eliminating just $k$ edges [i. e. one from each 3-component rising nonplanarity] $G$ becomes free planar and reversely, $G + e + f_i$ is nonplanar, where $f_i$ is such edge from arbitrary

3-component that gives rise to nonplanarity that violates this feature. Thus, only in the case when $k = 1$ we get something resembling directly $\mathcal{M}$-class's main feature that eliminating to edges we cross some free class. But this may occur on arbitrary large graphs and arbitrary large 3-components. Because of this *Planar* is actually $\mathcal{M}$-class.

Further some examples of $\mathcal{M}$-classes. $N_\circ(\{K_{3,3}\})$ is an $\mathcal{M}$-class because proper vertex split operation is not possible. In fig. **??** the forbidden graphs of a non-trivial example of $\mathcal{M}$-class are given. Fig. 4 shows instances of these forbidden graphs after proper vertex split operation is applied to them. Fig. 5 shows the instances of the forbidden graphs of corresponding free class of this $\mathcal{M}$-class where proper vertex split operation is possible. Fig. 6 shows instances of the graphs of Fig. 5 where proper vertex split operation is applied to them.

One more proof of Kuratowski theorem follows.

**Theorem 0.14** (Kuratowski)**.** *The class Planar is equal to* $N_\circ(\{K_{3,3}, K_5\})$.

*Proof.* Let us assume that there is some forbidden minor $H$ for class *Planar* distinct from $K_{3,3}$ and $K_5$ and it is 3-connected and minimal. Then neither $K_{3,3}$ nor $K_5$ are minors of $H$.

Let $v \in V(H)$ $deg(v) = k > 3$. Let $v$ be split arbitrary in $v_1$ and $v_2$ in such a way that $deg(v_1) = 2$ and $deg(v_2) = k-2$, thus obtaining graph $H_1$ from $H$. Let edges $s_1v_1$ and $t_1v_1$ be incident with $v_1$ in $H_1$. Let us choose such split of $v$ that $H_2$ is nonplanar of higher genus than before this split. Such possibility must be always present because $H$ neither has hinges nor is equal to $K_5$. But $H_1 - v_1t_1$ must be planar because $H - vt_1$ is planar [because of minimality of $H$]. But one edge $[v_1t_1]$ can not cause higher nonplanarity [than genus one of $H_1$] from state of planarity [of $H_1 - v_1t_1$]. Contradiction. Thus, if we assume $H$ to have vertices of degree higher than three then we come to contradiction.

Then $H$ must be cubic graph and can not have $K_5^-$ as minor. Then for arbitrary edge $e$ of $H$, $e = vw$, and arbitrary cycle $c$ in $H-e$ through $v$ and $w$ there must be two chords with respect to $c$ that prevent against embedding $c$ with both chords and $e$ in the plane. They form together $K_{3,3}$. We have come to contradiction that $H$ does not have $K_{3,3}$ as a minor. ☐

The proof of Kuratowski theorem in [4] is improved in [6].

**Acknowledgements.** I would like to thank Jan Kratochvíl for invaluable help to prepare this note.

pt]fig1.lpNon-trivial example of $\mathcal{M}$-class: Forbidden graphs of an $\mathcal{M}$-class $A$ with one forbidden graph from $\mathcal{G}(5,8)$, two forbidden graphs from $m(6,7)$ and one forbidden graph from $\mathcal{G}(7,6)$.fig1

FIGURE 3. 1

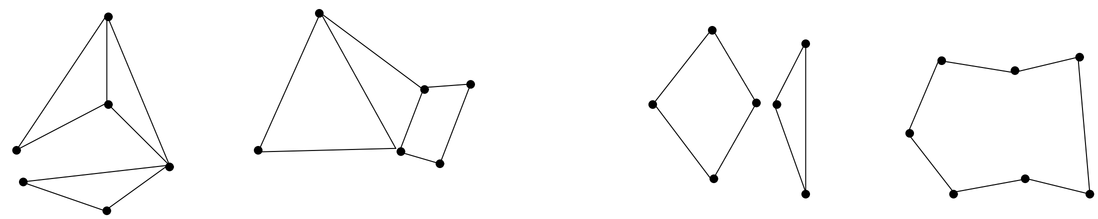

FIGURE 4. Non-trivial example of $\mathcal{M}$-class: Instances of the forbidden graphs of $A$ from fig. ?? after proper vertex split operation is applied to them.

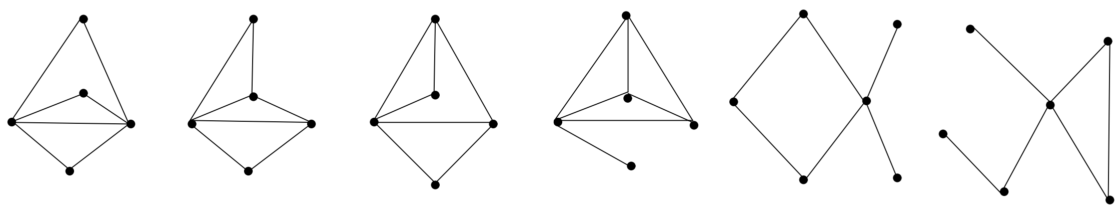

FIGURE 5. Non-trivial example of $\mathcal{M}$-class: Instances of the forbidden graphs of $Free(A)$ where proper vertex split operation is possible.

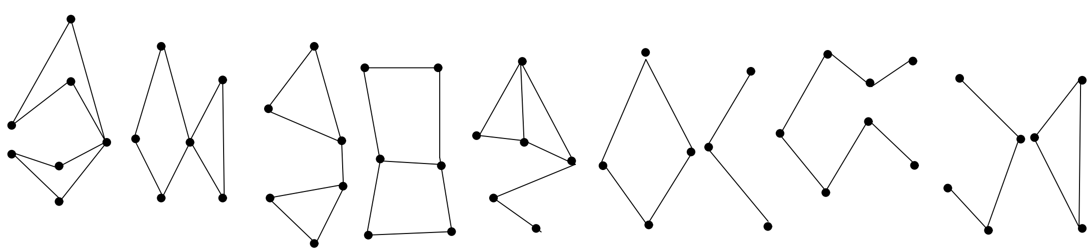

FIGURE 6. Non-trivial example of $\mathcal{M}$-class: Instances of the graphs from fig. 5 with proper vertex split operation applied to them.

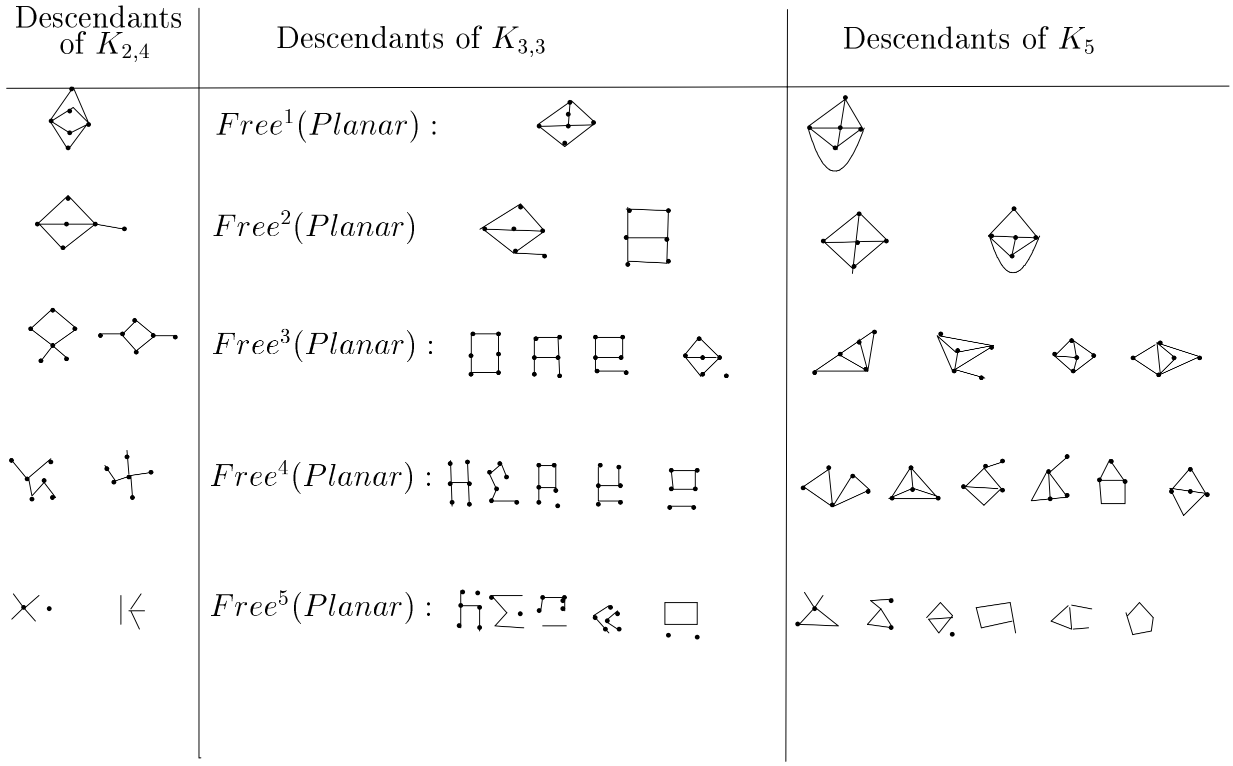


FIGURE 7. In the first column there are elements of $B(6,9)$ that have minors $X$ or $Y$ and do not contribute to $Free^k(Planar)$, $1 \leq k \leq 5$. In the second and third columns there are elements of $Free^k(Planar)$, $1 \leq k \leq 5$.

## REFERENCES


1. C. Chartrand, L. Lesniak. *Graphs and digraphs*, Chapman & Hall, 1996.
2. J. Kratochvíl *About minor closed classes and the generalization of the notion of free-planar graphs*, unpublished note, 1994, 2pp.
3. D. Zeps. *The Triconnectivity Considered Dinamically*, KAM Series N 90-168, KAM MFF UK, Prague, 1990, 6pp.
4. D. Zeps. *Free Minor Closed Classes and the Kuratowski theorem*, KAM Series, DIMATIA, 98-409, KAM MFF UK, Prague, 1998, 10.pp.
5. D. A. Zeps. *On the Dynamic Partitioning of the Graph in Threeconnected Components*, Republican Fund of Algorithms and Programs, Inv. N. UA0003Riga, 1984, 16pp.
6. D. Zeps. *Kuratowski Theorem from below*, Acta SML, 3rd LM Conf., Jelgava, 2000.

Author's address: Institute of Mathematics and Computer Science, University of Latvia, 29 Rainis blvd., Riga, Latvia.
*E-mail address*: dainize@cclu.lv