Graphs with rotations: Partial maps

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Abstract

Partial maps are considered.

1 Introduction

Since the note [2] of Edmonds, combinatorial maps have been studied by many authors [1, 3, 4, 6, 7, 8, 9]. In our approach [11] new text: Using permutations as a tool for combinatorial map, we can get a one-one map between permutations and graphs on surfaces. One way, how to do this, is shown in [11]. A way, how to exploit this, is for a chosen feature of permutations to find the corresponding one in the graphs. Reversely, if we can find for operations on graphs corresponding operations on permutation, then we can hope, that some nontrivial manipulations on graphs can be done using the simplest operations on permutations, i.e. multiplication, selection of submaps etc. In this work we argue to have proved a simple conjecture on permutations with less trivial graph-topological conjuncture in correspondence, but the main stress putting on the possibility to do corresponding graph-topological calculus using simple operations with permutations.

Other investigators [9]

Fixing a knot (in [1] called zigzag-walk)

2 Permutations and combinatorial maps

Permutations act on a universal set C the elements of which we call corners because of their geometrical interpretation. For a permutation P and $c \in C$ c^P denotes the element of C in which P maps c. Permutations we multiply from left to right. P^Q denotes $Q^{-1} \cdot P \cdot Q$ and is the conjugate permutation to P (with respect to Q). When S is a cycle $(c_1 c_2 \ldots)$ in P, then S^Q is the cycle $(c_1^Q c_2^Q \ldots)$ in P^Q. e denotes identical permutation.

In general we use the same terminology for combinatorial maps as in [11].

A pair of permutations (P, Q) we call *combinatorial map* whenever $P^{-1} \cdot Q$ is a matching, i.e. involution without fixed elements. The main characteristics of combinatorial map is its edge-rotation ϱ equal to $Q \cdot P^{-1}$ and next-edge-rotation π equal to $P^{-1} \cdot Q$.

Usually we are working with classes of maps with fixed π, that are closed under multiplication of maps from left.

3 Paths and cycles in maps

For a combinatorial map (P, Q), a sequence of elements $c_1, \ldots, c_n n > 0$ is *a path*, if for $c_i (0 < i < n)$ next element c_{i+1} in the sequence is equal to c_i^P or c_i^Q. If this same holds for c_n and c_1 also, i.e. c_1 equals to c_n^P or c_n^Q, then the path is closed, forming *a cycle*. *Simple* cycles are cycles without repetition of elements. For a combinatorial map simple cycles are transitive permutations on some subsets of the set C with the required low for chose of next element observed. *A cycle cover* of the combinatorial map (P, Q) is a permutation acting on the whole set C, where each

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of its orbits is a cycle in (P, Q). Then cycles in cycle covers are always simple cycles. Simplest cycle covers of (P, Q) are P and Q themselves.

Trivially enough, multiplying a cycle cover of some combinatorial map with some submap of its $\pi(= P^{-1} \cdot Q)$, we get another cycle cover of this combinatorial map.

We say, that two cycles ζ_1 and ζ_2 in (P, Q) touch each other when for same element e_1 of ζ_1 e_1^P or e_1^Q or e_1^{-P} or e_1^{-Q} belongs to the cycle ζ_2. Let us suppose, that the elements of C for some combinatorial map (P, Q) with a fixed cycle cover are colored in such a way, that 1)elements of cycles of the cycle cover are colored in the same color, and 2) cycles with equal element coloring do not touch. Such a coloring of elements of combinatorial map we call *a cycle cover coloring.*

4 Two-colorable cycle covers

Firstly let us consider two-colorable cycle covers generally.

Theorem 1 *Let us suppose, that for a combinatorial map (P, Q) with cycle cover ζ elements are colored in two colors, so that this is also a cycle cover coloring. Let an arbitrary edge be c_1, c_2, c_3, c_4 with its (possibly not all distinct) corners, such that $c_2 = c_1^{-P}$, $c_3 = c_1^{\pi}$ and $c_4 = c_1^{-Q}$.*

Then there are three possibilities:

1) c_1 and c_2 belong to the same color and c_3 and c_4 to the other color , [then we call such an edge cut-edge*];*

2) c_1 and c_4 belong to the same color and c_3 and c_3 to the other color, [then we call such an edge cycle-edge*];*

3) all corners of the edge are of the same color,[then we call such an edge inner edge*].*

Proof An edge is either a place where cycles of different color touch, or the same cycle meets itself (going either along the edge[like cycle-edge] or across the edge[like cut-edge]).

Let us notice, that , when the edge is not inner edge, the pairs (c_1, c_3) and (c_2, c_4) contain corners of different coloring. So, edges of one color in π and ϱ are inner edges, but with different color corners - cycle-edges and cut-edges.

For further considerations most useful are cycle cover colorings without inner edges. Then immediatelly is right what follows.

Theorem 2 *Let us suppose, that for a combinatorial map (P, Q) with cycle cover ζ elements are colored in two colors, green and red, so that this is also a cycle cover coloring and there are not inner edges.*

Then $| C_{green} | = | C_{red} |$, where $C_{green} \cup C_{red} = C$ and
π and ϱ are one-one matches between C_{green} and C_{red}.

Further we speak about the way how to get two-colorable cycle covers without inner edges in combinatorial maps, using a knot of this map. Combinatorially, knot is a zizag walk cover in the combinatorial map [1] and [11]. Zigzag walk always have orbits of even degree, so it is possible to connect with a zigzag walk a coloring of elements of combinatorial map. More over, elements of one color form cycles of one color, resulting in the cycle cover with coinciding coloring.

Theorem 3 *Let for a combinatorial map (P, Q) is given coloring $C_{green} \cup C_{red}$, and a knot of this map by this same coloring is colored alternatively. Then there exists one unique cycle cover of this map and the coloring is also this same cycle cover coloring without inner edges.*

Proof Let us suppose the opposite and elements of one color do not form cycles. Then there must be an element c, say of green color, but the colors of both $c^P(= c_1)$ and $c^Q(= c_3)$ are red. But the pair (c_1, c_3) belongs to π and its elements must have different colors. We come to contradiction. Trivially, there can not be inner edges.

Inversely, each cycle cover fixes some knot with precision to the reverse.

Let for a chosen μ of (P,Q) the corresponding cycle cover is ζ and let us express it as $\zeta_{green} \cdot \zeta_{red}$, where ζ_{green} contains cycles of green elements, and ζ_{red} contains cycles of red elements.

Theorem 4 *1) $\zeta \cdot \pi = \zeta_{altern}$ is a cycle cover with alternating coloring of its elements;*
2) $\zeta^{-1} \cdot P = \zeta_{altern}^{-1} \cdot Q = \pi_{cycle}$, where π_{cycle} have cycle-edges;
and $\zeta_{altern}^{-1} \cdot P = \zeta^{-1} \cdot Q = \pi_{cut}$, where π_{cut} have cut-edges;
and π_{cycle} and π_{cut} are complementary involutions and thus $\pi_{cycle} \cdot \pi_{cut} = \pi$.

5 Topology

Now we may think in terms of graphs and topology, but corresponding manipulations do in maps, i.e. permutations.

Let us partition C into $C_{cycle} \cup C_{cut}$, where the first set contains elements of π_{cycle} and the second - of π_{cut}.

Theorem 5 *The partial map $(P,\zeta) \mid_{C_{cycle}}$ is a combinatorial map on C_{cycle}.* Let us write ζ_{cycle} in the place of $\zeta \mid_{cycle}$, and so also ζ_{cut} in the place of $\zeta \mid_{cut}$.

Applying the theorem dually, we get, that also (Q,ζ) restricted on C_{cut} is a combinatorial map on this set.

We call $(P,\zeta) \mid_{C_{cycle}}$ and $(P,\zeta) \mid_{C_{cut}}$ correspondingly a *cycle graph* and a *cut graph.* The second one is in general a partial map.

Theorem 6

$$\gamma_{(P,Q)} > \gamma_{(P,\zeta)} + \gamma_{(Q,\zeta)}.$$

Proof Direct use of Eiler formula gives that there must hold an iniquality

$$\parallel \zeta \cdot P^{-1} \parallel + \parallel \zeta \cdot Q^{-1} \parallel + 2 \parallel \zeta \parallel > 3/2 \cdot l + 2c_{(P,\zeta)} + 2c_{(Q,\zeta)} - 2$$

, i.e. $\parallel \zeta \parallel > c_{(P,\zeta)} + c_{(Q,\zeta)} - 1$; where $c_{(P,\zeta)}$ and $c_{(Q,\zeta)}$ are number of components in the corresponding maps.

Further we try to show that this rather simple fact about permutations causes less trivial consequence in grafical-topological view.

6 Cutting of cycles in maps

From the graph topological point of view, when we cut a surface, in which a graph is embedded, along some cycle, then the surface is either cut to two parts or its genus is reduced depending whether the cut line along the cycle is contractable in the topological sence to point or not. Choising some cycle in cycle cover and cutting orbits in P along this cycle do the same thing in the combinatorial maps.

Let us denote multiplication $P \cdot P_{cycl}^{-1}$ by P and consider it nearer, seeing behind the partial map (P,Q) one with cut embendance surface along the cycles of cycle cover.

Theorem 7 *P, i.e. $P \cdot P_{cycl}^{-1}$ is equal to $\zeta \cdot \zeta_{cycl}^{-1}$ and genus of (P,Q) is equal to genus of (ζ, Q).*

Proof First we prove some lemmas.

Lemma 1 *Orbites of P_{cycl} has an alternating coloring, but orbits of P have elements of one color.*

Lemma 2 $P_{cycl} \cdot \pi_{cycl} = \zeta_{cycl}$.

Proof Before it is shown that $P \cdot \pi_{cycl} = \zeta$. Restricting this expression onC_{cycle} we get $(P \cdot \pi_{cycl}) \mid_{cycl} = \zeta_{cycl}$, but the left side is equal to $(P_{cycl} \cdot \pi_{cycl}$, what was to be proved.

Let us prove that $\zeta \cdot \zeta_{cycl}^{-1} = P$.

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