

Combinatorial map as multiplication of combinatorial knots

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Abstract

We show that geometrical map can be expressed as multiplication of combinatorial maps, i.e. map P is equal to multiplication of its knot, inner knot's square and trivial knot ($= \mu \cdot \nu^2 \cdot \pi_1$).

1 Introduction

We proceed with building combinatorial map theory that from different points of view and formulations is considered in from [1] to [35].

We multiply permutations from left to right. Geometrical *combinatorial map* is pair of permutations, *vertex and face rotations*, (P, Q) acting on set of elements C if $P \cdot Q^{-1} = \rho$ *edge rotation* or *inner edge rotation* $\pi = Q^{-1}P$ is involution without fixed elements. We consider set of maps with fixed π calling them *normalized maps*. Mostly we use one particular choice of π equal to (12) ... $(2k - 1 \ 2k)$, $k > 1$. If so, map may be characterized with one permutation, say vertex rotation P .

In [30] we saw that particular choice of ρ by fixed π induces partitioning of the set C into to subsets C_1 and C_2 [in general in several ways] so that the *knot* $\mu = \begin{cases} C_1 : \pi \\ C_2 : \rho \end{cases}$ is defined. Here, knot μ as permutation has 2^k choices if k is number of cycles in it. Rightly, changing direction of some cycle of μ we get another possible value for knot μ . Moreover, ρ with choice of particular μ partitions π into $\pi_1 \cdot \pi_2$, where we call π_1 *cut edges* and π_2 *cycle edges*, so that $P \cdot \pi_1 : C_1 \mapsto C_2$ and $P \cdot \pi_2 : C_1 \mapsto C_1$. In [31] was shown that by fixing μ map P may be expressed as multiplication $\gamma_1 \cdot \gamma_2 \cdot \pi_2$, where γ_1 acts within C_1 and γ_2 acts within C_2 .

In [30] was shown that normalized map always may be expressed as $P = \mu \cdot \alpha$, where α is called *knotting* and it is selfconjugate map in sense that $\alpha^\pi = \alpha$. In [32] we got formulas for μ and α , i.e.,

$$\mu = \gamma_2 \pi \gamma_1^{-1}$$

and

$$\alpha = \gamma_1 \gamma_1^\pi.$$

From [30] we know that α 's form a group K_π with respect to multiplication of maps. Moreover, classes of maps with fixed ρ 's, denoted as K_ρ , are cosets (left and right) of K_π .

2 Main part

We are going to regain main formulae from introduction.

Let us prove some theorems that leads us to the main result.

Theorem 1. $\rho \cdot \pi$ [or $\pi \cdot \rho$] is equal to some combinatorial knot μ squared and one or other color cycles induced from this knot reversed.

Proof. Let us write knot μ in the form $\begin{cases} C_1 : \pi \\ C_2 : \rho \end{cases}$. Then square of μ we would get applying $\pi \cdot \rho$ for one color corners and $\rho \cdot \pi$ for other color corners. \square

Theorem 2. By fixing the square of the knot it has 2^k knots in correspondence [in general for different maps] where k is the number of cycles in the knot.

Proof. Two joined cycles of square of knot may be combined in the cycle of knot in two ways, and thus, k independent operations give 2^k results. \square

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Theorem 3. [$\mu \cdot \pi$ is knot's half-square]

1) $\mu \cdot \pi$ contains squared knot's cycles of only one color.

2) For vertex rotation $\mu \cdot \pi$ corresponding face rotation and knot are equal to μ , and knotting equal to π . $\gamma_1 = id$, $\gamma_2 = \mu \cdot \pi$, and $\pi_1 = \pi$, because all edges of this map are cut edges.

Proof. 1) μ expressing as $\begin{cases} C_1 : \pi \\ C_2 : \rho \end{cases}$, and multiplying by π , we get $\begin{cases} C_1 : \pi \cdot \pi \\ C_2 : \rho \cdot \pi \end{cases}$ and using theorem 1 what was to be proved.

2) Corresponding graph to this map is set of star graphs as many as cycles in μ . Direct calculation gives what is stated by theorem. \square

Theorem 4. Map P can be expressed as $P\pi_1 = \gamma_1\gamma_2\pi = \gamma_2\gamma_1\pi$ with $\mu(P) = \gamma_2\pi\gamma_1^{-1} [= \gamma_1\rho\gamma_2^{-1}]$ and π_1 as inner cut edge rotation and π_2 inner cycle edge rotation.

Proof. Let knot $\mu = \mu(P)$ be fixed. Then set of corners is partitioned into two sets C_1 and C_2 . From form of $\mu (= \gamma_2 \cdot \pi \cdot \gamma_1^{-1})$ we directly judge that γ_1 belongs to, say, C_1 and γ_2 to C_2 . Thus, γ_1 and γ_2 commute by multiplying. Let us choose vertex rotation with this fixed knot and $\pi_1 = id$, i.e., with all edges being cycle edges. Then vertex rotation is alternation of corners from C_1 and C_2 respectively, and face rotation's cycles are correspondingly of one color. Then form of $\mu = \gamma_2 \cdot \pi \cdot \gamma_1^{-1}$ shows directly that $P \cdot \pi_1$ must be equal to $\gamma_1 \cdot \gamma_2 (= \gamma_2 \cdot \gamma_1)$. Finally, in general we get

$$\mu = \begin{cases} C_1 : \pi \\ C_2 : \rho \end{cases} = \begin{cases} C_2 : \gamma_2\pi\gamma_1^{-1} \\ C_1 : \gamma_1\rho\gamma_2^{-1} \end{cases}.$$

\square

Theorem 5. Map $P \cdot \pi_1$ can be expressed as $\begin{cases} C_1 : \beta_1 \\ C_2 : \beta_2 \end{cases}$, where involutions β_1 and β_2 are equal to $\beta_1 = \pi^{-\gamma_1}$ and $\beta_2 = \pi^{-\gamma_2}$. Moreover, $\beta_1 = \gamma_1\gamma_2^{-1}\mu$ and $\beta_2 = \gamma_2\gamma_1^{-1}\mu$. Moreover, $\beta_1\beta_2$ is squared knot $\mu(\dots, \beta_1)$ with one color cycles reversed. See theorem 1. $\delta = \pi^{\gamma_1}$. $P = \gamma_1\gamma_2\pi_2$.

Proof.

$$P\pi_1 = \begin{cases} C_1 : \beta_1 \\ C_2 : \beta_2 \end{cases} = \begin{cases} C_1 : \gamma_1\pi\gamma_1^{-1} \\ C_2 : \gamma_2\pi\gamma_2^{-1} \end{cases} = \left(\begin{cases} C_1 : \gamma_1 \\ C_2 : \gamma_2 \end{cases} \right) \cdot \pi = \gamma_1\gamma_2\pi.$$

\square

Corollary 6. Map $P\pi_1$ is a knot for inner edge rotation β_1 and edge rotation β_2 .

Theorem 7. Let for some fixed knot the map P be equal to $\mu\alpha$. Then α is equal to $\gamma_1\pi\gamma_1\pi_2$ or $\gamma_1\gamma_1^\pi\pi_1$.

Proof. $\alpha = \mu^{-1}P\pi_1 = \gamma_1\pi\gamma_2^{-1}\gamma_2\gamma_1\pi = \gamma_1\pi\gamma_1\pi = \gamma_1\gamma_1^\pi$. This α is knotting for $P\pi_1$. For map P knotting is $\gamma_1\pi\gamma_1\pi_2$ or $\gamma_1\gamma_1^\pi\pi_1$. \square

Corollary 8. $\alpha^\pi = \alpha$. Selfconjugate α 's comprise group.

Proof. $\alpha^\pi = (\gamma_1\gamma_1^\pi\pi_1)^\pi = \gamma_1^\pi\gamma_1\pi_1 = \gamma_2\gamma_1\pi_1 = \alpha$. \square

Theorem 9. $\gamma_1\gamma_1^\pi$ is some knot's square.

Proof. Let us denote this knot by ν . Direct observation shows that theorem is correct. Then fixed knot μ induces α and it determines fixed ν such that $\nu^2 = \gamma_1\gamma_1^\pi$. \square

Theorem 10. Every combinatorial P can be expressed as multiplication of knots in the form

$$P = \mu \cdot \nu^2 \cdot \pi_1.$$

Proof. It directly follows from previous theorems. Really, $P = \mu\alpha = \gamma_2\pi\gamma_1^{-1}\gamma_1\gamma_1^\pi\pi_1 = \gamma_2\gamma_1\pi_2 = \mu(\gamma_1\pi)^2\pi_1 = \mu\nu^2\pi_1$. \square

It must be noted that π_1 is some knot too. We call this knot trivial knot. Let us call knot ν map's inner knot.

Corollary 11. Map is multiplication of its knot with its inner knot's square and with its trivial knot.

Theorem 12. For $P\pi_1$ μ commutes with α , i.e.,

$$P\pi_1 = \mu \cdot \alpha = \alpha \cdot \mu.$$

In general,

$$P = \mu\alpha = \alpha\mu^{\pi_1}.$$

Proof. For $P\pi_1$, $\mu^\alpha = (\gamma_2\pi\gamma_1^{-1})\gamma_1\pi\gamma_1\pi$. Further, $\gamma_2^{\gamma_1\pi\gamma_1\pi} = \gamma_2^{\pi\cdot\pi} = \gamma_2$, because corners of γ_1 and γ_2 do not intersect. The same is true for the member γ_1^{-1} . Further, $\pi^{\gamma_1\pi\gamma_1\pi} = \pi^\alpha = \pi$. Thus, we get $\mu^\alpha = \mu$. \square

Theorem 13. For partial map $[P, \mu]$ its inner edge rotation is α .

Proof. Direct observation. \square

3 Conclusions

There are four types of permutations that are used to build "all" in combinatorial map theory, i.e., knot-type, knot-square-type, knot-square-with-reversed-cycles-type, two-color-involutions. Comprehensive algebra of all these types should be ground for combinatorial map theory.

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