About minor closed classes and some notion of freenness.

(A note given to Dainis Zeps in Spring 1994)

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Notation $G - e$ graph obtained by deletion edge e from G
$G - v$ graph obtained by deleting vertex v from G
$G \circ e$ graph obtained by contracting edge e
(picture)
$G \odot e$ any graph obtained from G by splitting vertex v (not unique!)
(picture)
$G \subset H$ G is a subgraph of H, i.e. $G \cong G'$ and $V(G') \subset V(H')$, $E(G') \subset E(H')$
$G \prec H$ G is a minor of H, i.e. G can be obtained by edge contractions from a subgraph of H, i.e. G can be obtained by vertex deletions, edge deletions and edge contractions from H

Definition A class of graphs A is called minor closed if
$\forall G, H : G \prec H \in A \Rightarrow G \in A$.

Notation For a minor closed class A, $F(A) = \lfloor \{G \mid G \notin A\} \rfloor$, here
$\lfloor B \rfloor = \{G \mid G \in B \wedge \forall H : (H \in B \wedge H \prec G) \Rightarrow H \cong G\} = \{\leftarrow$ minimal graphs in $B\}$.

Proposition For a minor closed class A, $\forall G : G \notin A \Leftrightarrow \exists H : H \in F(A) \wedge H \prec G$.

Theorem (Robertson,Seymour) $F(A)$ is finite for any minor closed A.

Definition $Free(A) = \{G \mid \forall e \notin E(G) : G + e \in A\}$

Proposition A minor closed $\Rightarrow Free(A)$ is minor closed.

Notation For a class A: $A^- = \{G - e \mid G \in A, e \in E(G)\}$
$A^{\odot} = \{H \mid H \cong G \odot u, G \in A, u \in V(G)\}$.
$N_\circ(A) = \{G \mid \forall H \in A : H \not\prec G\}$, i.e. $F(N_\circ(A)) = \lfloor A \rfloor$.

Theorem For a minor closed class A, $F(Free(A)) = \lfloor F(A)^- \cup F(A)^{\odot} \rfloor$

Proof We are proving $Free(A) = N_\circ(F(A)^- \cup F(A)^\circ)$

1) $\subset$: $G \in Free(A) \Rightarrow G \in N_\circ(F(A)^- \cup F(A)^\circ)$ i.e.
$G \notin N_\circ(F(A)^- \cup F(A)^\circ) \Rightarrow G \notin Free(A)$
$\exists H \in F(A)(H - e \prec G \vee H \odot v \prec G) \Rightarrow H \prec G + e \Rightarrow$

$\Rightarrow G + e \notin A \Rightarrow G \notin Free(A)$

2) $\subset$: $G \notin Free(A) \Rightarrow G \notin N_\circ(F(A)^- \cup F(A)^\circ)$
$\exists e : G + e \notin A \Rightarrow \exists H \in F(A) : H \prec G + e \Rightarrow$
$\Rightarrow H \prec G \vee H - f \prec G \vee H \odot u \prec G \Rightarrow$
$\Rightarrow G \notin N_\circ(\{H\}^- \cup \{H\}^\circ) \Rightarrow G \notin N_\circ(F(A)^- \cup F(A)^\circ)$.

Example 1. $A =$ Planar graphs
$F(A) = \{K_{3,3}, K_5\}$
$F(A)^- = \{K_{3,3}^-, K_5^-\}$ $F(A)^\circ =$ (picture)
(picture) and hence
$\lfloor F(A)^- \cup F(A)^\circ \rfloor = \{K_{3,3}^-, K_5^-\}$. ∎

2. $A = N_\circ(K_5)$
$F(A) = \{K_5\}$
$F(A)^- = \{K_5^-\}$ (picture)

Theorem $F(Free(planar)) = \{K_{3,3}^-, K_5^-\} \Rightarrow$ Kuratowski theorem.
Proof ???
Bad example For $A = N_\circ(K_{3,3})$ and
$B = N_\circ$('two-samples-of-$K_{3,3}^-$-connected-with-edge', $K_{3,3}$),
$\lfloor F(A)^- \cup F(A)^\circ \rfloor = \lfloor (F(B)^- \cup F(B)^\circ \rfloor = \{K_{3,3}^-\}$ and
$Free(A) = Free(B) = N_\circ(K_{3,3}^-)$. !!!
Problem Do there exist graphs $G, H, G \ncong H$,
s.t. $\lfloor F(\{G\})^- \cup F(\{G\})^\circ \rfloor = \lfloor F(\{H\})^- \cup F(\{H\})^\circ \rfloor$?