Application of the Free Minor Closed Classes in the Context of the Four Color Theorem

Dainis ZEPS ⋆

Abstract. Four color theorem, using concept of free-planar graphs, is discussed.

1 Introduction

We have two main ideas and one observation in the ground of this treatment. Free planar graph idea arouse considering dynamic graph partitioning into 3-connected components [12, 13]. Secondly, free planar graphs were generalized in free minor closed classes in [7] by Jan Kratochvíl. Treating in year 1984 graph's dynamic partition into 3-connected components we observed that this partitioning algorithm can't be simplified if we had intent to build only free-planar graphs. Thus, speaking in terms of this observation, planar graphs are not more complicate than free-planar graphs, at least what concerns their re-constructibility from 3-connected components [12]. Thus, algorithmically or what concerns algorithmic re-constructibility of the graphs, planar graphs are not more complicate than free-planar graph in spite of fact that free-planar graphs are only small subset of planar graphs. Further, from four-color theorem we know that planar graph uses the same number of colors to be colored as free-planar graph. In this article we investigate how this simple fact could be put in the ground of some other approach in trying to prove four-color theorem.

Further in this article we directly mention either four-color theorem is assumed already proved or not. When it is not alleged directly then it should follow from the context.

We start with the proof what we believe should be put to ground for four-color theorem in general.

Theorem 1. *Every free planar graph is colorable using four colors.*

Proof. Direct check shows that 3-connected free planar graphs are 4-colorable graphs [15]. There are only three cases to be considered. Wheel graph is 4-chromatic if odd, and 3-chromatic when even. Envelope graph is 3-chromatic. Further, uniting 3-components of free planar graphs, which are at most 4-colorable graphs, via virtual edges, we get again at most 4-colorable graphs. Truly, virtual edge itself may acquire no more than two colors, what is the maximal number of colors that may be forced upon an another component. Here we must recall that 3-components which are unique comprise 3-component tree, i.e., structure without cycles [12, 13].

2 Free-k-chromatic graphs

Graph is free-k-chromatic if it is k-chromatic and for every non-edge e $G+e$ is k-chromatic too. More general, graph is n-free-k-chromatic if it is k-chromatic and for every non-edge set of n edges G with these edges added remains k-chromatic. Shortly we denote this

⋆ Author's address: Institute of Mathematics and Computer Science, University of Latvia, 29 Rainis blvd., Riga, Latvia, dainize@mii.lu.lv

property by 'nfk', saying, for example, that graph G is 2f4-graph, meaning that it is 2-free-4-chromatic graph.

Let us try to clarify how freeness of colorability could be connected with freeness of minor closed classes. One simple result we get directly.

Theorem 2. *Every k-critical graph is free-k-chromatic*

Proof. Every edge may receive the lonely color and thus every non-edge may be added non disturbing coloring.

Further, we may conclude that class of free-k-chromatic class is wider that k-critical class. For example, graph K_{k+1} with proper splitting of vertex is free-k-colorable graph, but not k-critical graph. What concerns 4-colorability, one more large class belongs to free-4-chromatic graphs, and this is class of free-planar 4-chromatic graphs. This follows from the four color theorem. Actually, if we add edge to such graph, it should remain planar and, thus, 4-chromatic according four color theorem.

But we want to prove this fact not using four color theorem.

Actually, k-chromatic free planar graphs with $k < 4$ can't rouse any problems what concerns four-color theorem. Truly, we may apply for new edge always fourth color. We go directly to case $k = 4$.

But, before we need to prove simple theorem.

Theorem 3. *Every atetrahedral graph is 3-colorable, and, if 3-chromatic, graph is free-4-chromatic graph.*

Proof. Atetrahedral graph itself is 3-colorable by induction: start coloring with odd cycle as 3-component, using 3 colors; then after, as step of induction, add some connected component to already colored part. If all cycles are even than graph should be 2-chromatic. It is trivial that 3-chromatic atetrahedral graph is free-4-colorable.

Theorem 4. *Every free planar 4-chromatic graph is free-4-chromatic graph.*

Proof. If graph is 3-connected than we must check only case of odd wheel graph: adding edge it should remain 4-chromatic, because odd wheel graph is 4-critical graph, see previous theorem. Further, we must show that arbitrary free-planar graph consisting from more than one 3-component, where at least one component is odd wheel, remains 4-chromatic after edge is added.

Let graph be 4-chromatic free-planar graph consisting from more than one 3-component. Let us add edge $x - y$ to check whether graph remains 4-chromatic.

If both ends of edge, i.e., vertices x and y, fall in the same 3-component, then we have all proved. Edge $x - y$ is going to contract path of 3-components, see [12, 13] (or path of g-edges [16]). We may refer problem of coloring the contractible part of graph to problem of coloring atetrahedral graph plus edge. But atetrahedral graph is 3-colorable, as we have this proved already in the theorem before, and proof of theorem is completed.

We have proved that four color theorem is true what concerns free planar graphs and even augmented with additional edge, i.e., that free planar 4-chromatic graph is free-4-colorable too. But this evidently is not sufficient to conclude about the four color theorem in general. Let us go further.

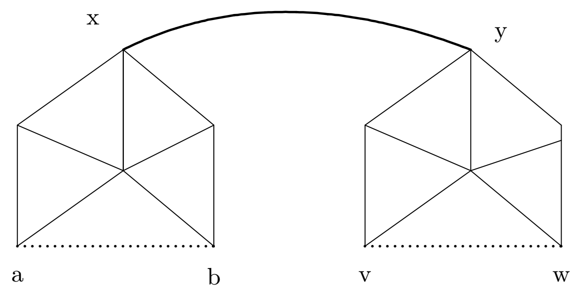

Fig. 1. Idea of free-four-color theorem for case of free-planar graphs. 3-components of free-planar graph may be replaced with 3-cycles or 4-cycles, for example, it suffices to consider triangles $\{x, a, b\}$ and $\{y, v, w\}$

Theorem may be extended. It is easy to see that in place of free planar graph class may be taken some wider class, let it be called class PP, i.e., the class generalizing free-planar graph class, without forbidden virtual edges in building graphs, contracting path of 3-components according procedure in [12]. Graphs from PP have the same components as free-planar with exception that forbidden virtual edges (for free-planarity) are now allowed. Class PP is expected to comprise free-planar (or even free-2-planar) graphs for projective planar , though at least one graph violates this expectation, D_{17}, forbidden graph for projective plane [5].

Let us define class PP precisely. Similarly as in [15], page 6, graph G from PP consists from components, where every edge may be virtual edge, as follows:

1) C_k, $k > 2$;
2) W_k, $k > 2$;
3) $\overline{C_6}$.

It is easy to see that class PP does not have free-Hadwiger class $Free(H_5)$ [where H_5 supposed without minor K_5] as subclass, because forbidden graph $K_5^{\odot}$ belongs to PP, but not forbidden $K_{3,3}$ doesn't belong to PP. Thus, two classes intersect. Next two theorems speak more precisely.

Theorem 5. *Class PP is minor closed class.*

Proof. Checking each component separately, we may see that eliminating edge or contracting edge we get some legal component. So, from W_k we get either some other wheel or polygons. From envelope graph $\overline{C_6}$ we get K_4 and polygons.

Theorem 6. *Class PP is equal to class $N_\circ(K_5^-, K_{3,3})$.*

Proof. If we added constraint as forbidden graph $K_{3,3}^-$, we had PP equal to free-planar graph class. Taking off constraints from virtual edges just means allowing minor $K_{3,3}$.

Further, K_5^- can't be obtained with allowed components. Neither $K_{3,3}$ can.

In first case, largest component in K_5^- minus edge is either W_4 or K_4, which already are closures for allowed set of components, i.e., can't be obtained from allowed 3-components with connections via virtual edges. In second case, only $K_{3,3}^-$ without closure with edge may be obtained.

Theorem 7. *Every graph from class PP is free-4-chromatic*

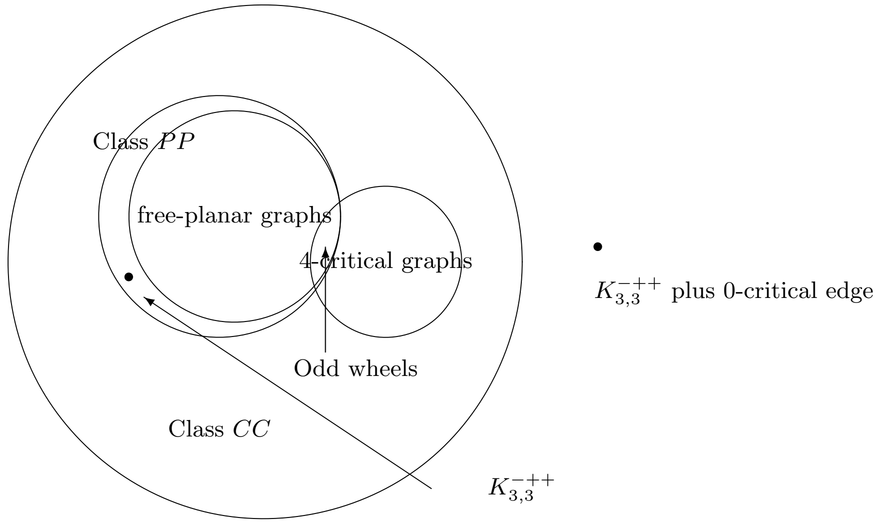

Fig. 2. Classes comprising all free-4-chromatic graphs. One graph, proper $K_5\odot$, or $K_{3,3}^{-++}$ has $K_{3,3}^{-}$ as subgraph, and belongs to class PP, but not belongs to $Free(H_5)$. Graph $K_{3,3}^{-++}$ plus 0-critical edge remains to be 4-critical, but does not belong to any class in the picture, except $Free^4CH$ itself. Graph proper $K_5^{\odot} = K_{3,3}^{-++}$ especially should be taken into account because it is the only graph that is free-4-chromatic with 0-critical edge and would remain free-4-colorable after this 0-critical edge be added. Class $Free(H_5)$ that would be between classes $Free(Planar)$ and CC and intersect with class PP is not shown in the picture.

Proof. In proof of theorem 4 we may observe that it does not depend upon fact either virtual edges have constraints as in case of free-planar graphs or not. Thus previous proof fits for this theorem too.

It is easy to see that the proof of the last theorem would not change if in place of odd wheels we had arbitrary 4-critical graph. Let us use this fact and with purpose to widen the observable class of free-4-chromatic graphs define class CC which is modified class PP with 4-critical graphs allowed in place of odd wheels. Then we formulate as corollary what follows, because previous proof fits as before.

Corollary 8. *Every graph from class CC is free-4-chromatic.*

Let us denote by $Free_kCH$ class of all free-k-chromatic graphs. We may conclude the following corollary.

Corollary 9. $Free(Planar) \subset Free(H_5) \subset \langle K_{3,3} : K_5^{\odot} \rangle \supset PP \subset CC \subset Free_4CH$, *where* $\subset \langle G_1 : G_2 \rangle \supset$ *denotes symmetric property saying that classes intersect with at least G_1 not belonging to right part, and G_2 not belonging to left part.*

Here we have one exceptional graph, proper $K_5^{\odot}$ plus non-planar edge, which does not fit in any of classes above, but it is free-4-chromatic. This same graph may be imagined as $K_{3,3}^{-++}$, i.e., $K_{3,3}$ minus edge plus two edges not affecting planarity.

0-critical edge is such non-edge in graph that for all colorings it may be added to graph no affecting these colorings. 1-critical edge then is such non-edge that for no coloring it may be added without affecting total coloring of the graph. Addition 1-critical edge to graph raises graphs chromatic number.

Lemma 10. *Graph $K^{\odot}_{k+1}$ with split being proper, i.e., not simply edge being split off, is free-k-chromatic with 0-critical edge, that corresponds to split vertex. More over, it remains free-k-chromatic after edge vw is added.*

Proof. Let new vertices in the split would be v and w. It is easy to see that vw is 0-critical edge, because vertex before split had k+1-th color, but after split v and$w$ should receive their colors from mutual neighbors, i.e., different colors. Every edge may be added to this graph; if, say, color of v is the same that some, say, vertex s, then s may change its color with other vertex from neighbors of w. vw may be added as well. The act of adding does not make graph not being free-k-chromatic, as it follows from previous arguments.

Conjecture 11. *Graph proper $K^{\odot}_5$ is the only free-4-chromatic graph with 0-critical edge.*

Proof. Let us have some considerations in favor of this conjecture. Let us consider at least these classes what we build here. We have to prove that other free-4-chromatic graphs do not have 0-critical edges. If added edge falls into single component, say, some k-critical graph, $k < 5$, then it is not 0-critical. If edge falls in different components then vertices via virtual edges can't be forced to have or not to have different color.

In other cases, only uniquely colorable graphs or subgraphs may force 0-critical edges, but we have opposite, free-colorable graph.

Graph $K^{\odot}_5$ is not the only graph that would belong to class $Free^2_4CH$, there at least one more such graph may be given, i.e. D_{17} from forbidden subgraphs of projective plane [5, 17]. Further we would see that all critical graphs behave the same way.

Let us 'configure' our graph proper $K^{\odot}_5$, or $K^{-++}_{3,3}$, in the sequence of subsets in corollary 9.

Corollary 12. *$K^{-++}_{3,3} \notin Free(H_5)$, $K^{-++}_{3,3} \in PP$, $K^{-++}_{3,3}$ plus 0-critical edge is not in PP, and $K^{-++}_{3,3}$ plus 0-critical edge is in $Free_4CH$.*

Class CC hardly could be expected to be characterizable with forbidden components. But some more special case is possible. Graphs G_{2k+1}, with $k > 1$, from [18] that there are called higher order wheels, are 4-critical, does not have minor $K^{\odot}_5$ and thus this class is subclass of class $N_{\circ}(K^{\odot}_5, K_{3,3})$. If we would like to constrain our class CC to these 4-critical graphs then all class became subclass of this class.

By the way, these higher order wheels G_{2k+1} have O^-, where O is octahedron graph, as minor. But, O itself is forbidden graph for all class of G_{2k+1}, $k > 1$. Let us formulate it as nice corollary.

Corollary 13. *For the family of higher wheels octahedron minus edge is proper grandson and octahedron itself is proper grandfather.*

In [18] defined minor brackets allow to characterize this situation. For higher wheels pair (O^-, O) is proper minor bracket.

Let us add to class's CC, calling it completed CC, graph with as allowed components linear graphs l_k, i.e., paths of length $k - 1 > 0$, where each vertex may serve as virtual vertex.

Let us define class of graph $4M$ as follows: graph G belongs to $4M$ if its spanning subgraph belongs to completed CC.

Theorem 14. *If four color theorem is correct class $4M$ contains all planar graphs.*

Proof. Let graph be 4-colorable. Let us reduce edges until it is free-4-colorable. Graph belongs to $4M$.

Of course, we must be content with the fact that class CC can't be minor closed. Either whole class of free-4-colorable graphs can't be expected to be minor closed.

Theorem 15. *If four color theorem is true, in 4-chromatic planar graph edge to be added is either non-1-critical and planar or 1-critical and non-planar.*

3 Closures of components

In [16] we introduced notion of component path closure and proved Kuratowski theorem analogue for 3-components. Let us remind some definitions from there.

Components we discuss are the same as in theory of graphs dividing into 3-connected components, i.e., polygons, bonds [if necessary], three connected graphs, with as many virtual edges as necessary. In considering free-planar graphs, our allowed components should be these allowed in building free-planar graphs, and so on. We add as very useful notion generalized vertex and edge,or g-vertex and g-edge. g-vertex is either single vertex or two distinct vertices in the graph. If not told otherwise, we would assume that with g-vertex virtual edge is connected, i.e., if $\{v, w\}$ is g-vertex then $\{v, w\}$ is virtual edge in G too. In case g-vertex is single vertex, it is virtual vertex too. g-edge is always pair of g-vertices, i.e., if not told otherwise, pair of virtual edges, or edge or vertex, or virtual vertices. Thus, g-edge may consist from four vertices, or three vertices, i.e. vertex and edge, or two edges with common vertex, or two vertices, i.e., vertex and vertex.

We are saying that component C is marked with g-vertex $v = a$ or $v = a, b$, saying $v : C$ is formed, when some g-vertex v is distinguished in C, and v is virtual vertex in C. Distinguishing two distinct g-vertices in C, we may speak about g-edge being distinguished in C, denoting it $vw : C$ or $e : C$, where vw and e is g-edge.

Let us call closure union two components by common g-vertex, that is simple vertex. For example, we may write $C_1 : e : C_1$, assuming that two marked components $e : C_1$ and $e : C_2$ have common g-vertex e, and say that components are forming closure by uniting two g-vertices in common g-vertex e. If order of e is one, i.e., it is simple vertex, then we call closure 1-closure, first order closure. If order of e is two, i.e., it pair of vertices, then we call closure 2-closure, or second order closure. Further, we may form paths of closures $C_1 : e_1 : C_2 : e_2 : ...e_{k-1}C_k$ with $k > 1$. Yet more, we may cycle in this path, speaking about new type of closure, i.e., if we cycle in path of first order closures, then we say that we did *2-closure from 1-closures*, and, if we cycle in path of second order closures, i.e., at least one closure in the path was second order, then we say, that we did *3-closure from 2-closures.*

3.1 Building 4-critical graphs

First order 4-chromatic higher order wheels In [18] higher order wheels were introduced, that were 4-critical graphs. We may now illustrate on these wheels in what way 3-closures from component 2-closures path may be very useful.

Figure 3 explains in example how closure of path of closures is used to build 3-connected graphs.

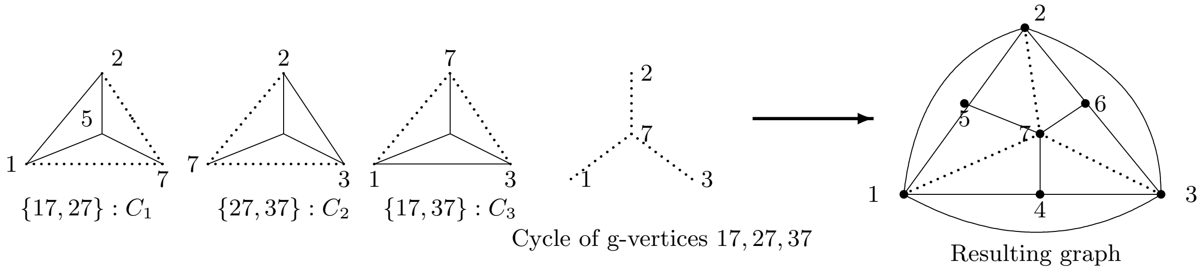

Fig. 3. Resulting graph is formed as 3-closure of path of 2-closures $C_1 : \{2,7\} : C_2 : \{3,7\} : C_3 : \{1,7\}$:. Actually, cycle of g-edges is $\{17,27\},\{27,37\},\{37,17\}$, but we may in place of it considering corresponding cycle of g-vertices $17,27,37$.

Let for a while us consider only wheels as components. Then graph performed from simple wheels as closure of path of closures may be characterized as sequence $n_1, ...n_l$, $l >= 3$, where n_i, $1 <= i <= l$, is order of i-th wheel.

Theorem 16. *Graph obtained as closure of path of wheels as components is 4-critical iff* l *is odd and every* n_i *in the characteristic sequence of the closure is odd and at least three.*

Proof. Let $l = 3$, and wheel be minimal. Common vertex in virtual g-edges in cycle of closures should receive colors $(A \vee B) \wedge (B \vee C) \wedge (C \vee A)$ which gives $Falce$, thus new color should be generated. Every edge in construction is crucial.

Let wheel be arbitrary. The same coloring as for simple case is now possible too. Every edge is crucial in the construction by the argument that follows. For absent rim edge, rim vertices may be colored with, say, colors A and B. For absent outer and inner spike non-spike's both ends colors with the same color.

Thus, chromatic criticality rises from the condition that every edge is necessary in the construction.

Case for arbitrary odd l generalizes obviously: condition for center of wheel could be made $(A \vee B) \wedge (B \vee A) \wedge (A \vee B)...(B \vee C) \wedge (C \vee A)$.

It is easy to see that, if in sequence $n_1, ...n_l$, $l >= 3$, at least one number not excluding l is not odd, then graph constructed may be colored with three colors.

In the article [18] graphs considered were $3-3-3$ (G_3), see figure 3, $3-3-5$ (G_5),$3-5-5$ (G_7), and $5-5-5$ (G_9), see figure 4.

Class of non-three-connected 4-critical graphs In [16] graph in fig. 6 was wheel graph W_5 with split one edge what may be considered as smallest non-free-planar 4-critical graph. Either it is example of smallest application of Hajós sum [2]. See fig. 9. This construction may generalized to give augmentable class of similar graph that would be non-tree-connected, but nevertheless 4-critical.

In place of two components K_4, there may be even number of K_4. It is easy to see that such graph should be 4-critical. It has general cycle of odd length. Next, on this cycle one vertex should receive 'third color', and, as a consequence, in that component fourth color should appear.

Graphs of this class are characterized by number of components K_4, divided by two, calling it graph's order. Thus, in fig.9 there are graphs of first and third order of the type.

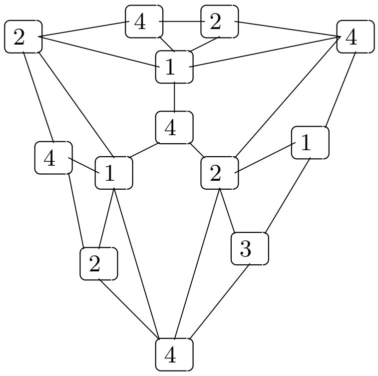

Fig. 4. Graph $5 - 5 - 5$ (G_9 in [18])

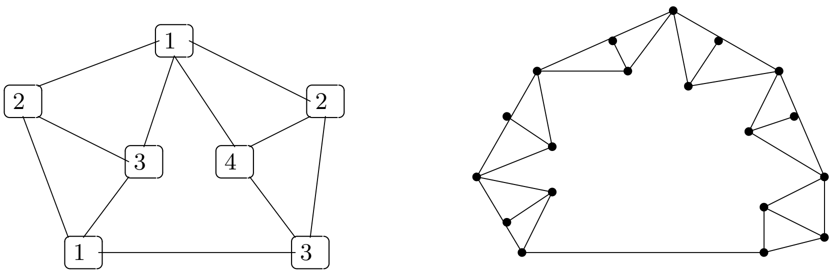

Fig. 5. Hajós construction repeated [2].

3.2 Closures of paths of closures

Further, components in paths in closures may be arbitrary, i.e., not only wheels.

Theorem 17. *Let A be arbitrary class of graphs which is generated from some set of allowed possibly marked components without yet applied closure of paths of closures, and graphs of this class are free-4-chromatic. Let us build new graph where one closure of path of closures is applied. Then the graph is free-4-chromatic too.*

Proof. Proof is omitted here.

It may be expected that previous theorem may be generalized to case when paths are built in a way when cycles only intersect but do not touch in sort of edges. Formally, it should be done carefully. It is clear that overall generalization with whatever possible closures should give graphs that are not free-4-chromatic. It is obvious. Because, using this technics we may construct arbitrary graphs, but not every graph is free-4-chromatic.

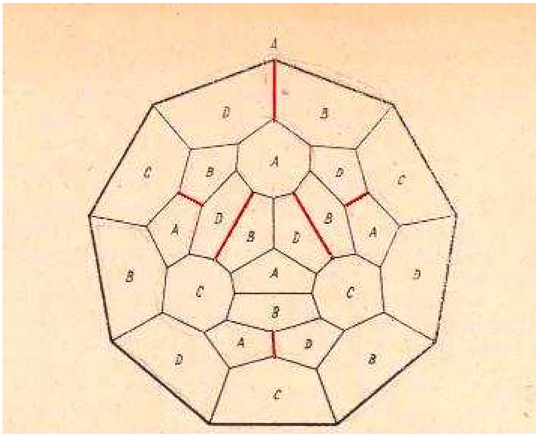

Fig. 6. Example of map with split [red] edges. Graph becomes free planar. In this case virtual edges should coincide with real edges.

3.3 Planar graph's reduction to free-planar graph

Pictures 9 and 11, using one Grinberg's graph [6], demonstrate way, how graph may be split in a way it becomes free-planar. In first case splits are done via edges, i.e., in that case, according Tutte's theory of division into three-connected components [11, 13], virtual bonds with one real edge and virtual edges should appear. In another case splits are done, no affecting real edges in the graph. Both types of splits may be combined together, but we would prefer second type of splits and use only them in order to simplify theoretical outline.

3.4 Another simple proof of four-color theorem. What's wrong with it?

Let us have planar graph G with m edges and one edge take out, and let us, under induction assumption, it's vertices color in four colors, and let us split it, using second

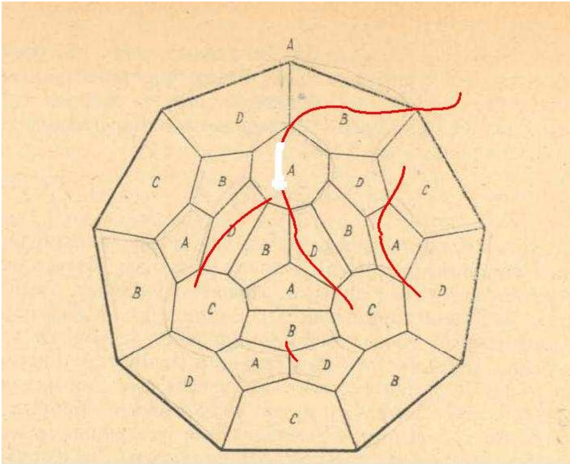

Fig. 7. Graph becomes free planar after four edge-type splits and one vertex-type split.

way, until it becomes free-planar, remembering vertex colorings. Now we have graph that is free-planar and free-4-chromatic. We may add eliminated edge, and graph is with m, and proof almost done.

What's wrong with this proof? Free-4-chromatic graph after adding edge should be, in general, recolored. The operation of recoloring may affect end g-vertices that should be conditioned by some border condition, according which split vertices or edges should preserve coloring, that is in concordance with coloring in graph without splits. If recoloring is possible with the preserving of this border condition, then proof would be done too. Let us say that graph may be colored in 4 colors if this border condition problem may be solved, and call it *border condition problem.* Let us formulate this as theorem.

Theorem 18. *Border condition problem may be solved iff four color theorem is correct.*

Formulating this theorem we want to express significant principal fact. What concerns number of colors, needed in coloring, *coloring planar graph is as complex as coloring free planar graph.* I.e., four colors are needed in both cases.

3.5 Discussion of border condition problem

How to attack border condition problem? We may show up two encouraging facts. First, in [6] Grinberg mentions significant fact that only possible counterexamples of 4CT may be quasi-five connected graphs. Thus, in the border condition problem this fact should play some role. Second, if in traditional proof outline of 4CT we must consider 5-critical graph, then in our case we should consider 5-critical free-planar graph with border condition. But graph is free-planar nevertheless, where only wheels play any significant role, thus, our 5-critical graph consist from wheels connected by polygons or envelopes and with border condition.

3.6 4CT proof

Let us start as before with induction assumption, but assuming that two edges are taken out and adding one, border condition problem is solved by induction assumption.

Let us consider the only critical situation, odd wheel, where two edges being absent, it colored using four colors, in general in several ways. Adding one edge we always solve border condition problem by induction assumption, in general with several possible colorings of the graph with $n-1$ edges. There remains one edge to be added. Both edges may be interchanged, by symmetry assumption. If wheel had $2k+1, k \geq 1$, spokes, then $4k(2k+1$ cases are possible. By the way, situation in general, as described above, has evident central symmetric symmetry. Is border condition problem the same, i.e., with the same symmetry? Seems not, but something like what could be expected in outer planar graph.

Let us go further. *In spe.*

4 n-free-k-chromatic graphs and unique colorable graphs

Free minor closed classes may be characterized by forbidden minors of classes against which class is made free. This is made via Kratochvil's theorem [7, 15].

In [7] Kratochvíl proved very powerful theorem :

$$F(Free(A)) = \lfloor F(A)^{-} \cup F(A)^{\odot} \rfloor,$$

where $B^{-} \doteq \{G - e \mid G \in B, e \in E(G)\}$ and $B^{\odot} \doteq \{H \mid H \cong G \odot v, G \in B, v \in V(G)\}$ and operation $\odot$ [in its application $G \odot v$] denotes a non unique splitting of vertex v in G, which is the opposite operation to edge addition and its contraction [in result giving vertex v].

We may try to characterize classes $Free_k CH$ similar as in case of free minor closed classes with forbidden minors, but now we expect not to use minors but subgraphs. Forbidden subgraph for 4-chromatic graph is K_5, for example, next to it arbitrary 5-critical graph, and, if four color theorem is not correct supposedly, let us say, then H, the smallest counterexample to four color theorem, that should be 5-critical graph, is similarly forbidden subgraph for 4-chromatic graphs.

Let us fix an almost trivial assertion as corollary and then generalize it as non-trivial theorem.

Corollary 19. *Graph G is k-chromatic iff it does not contain as direct subgraph k+1-critical subgraph and* $\chi(G) \not< k$.

Proof. If G has as direct subgraph k+1-critical graph, it can't be k-critical.

Let G be k-chromatic. Then it can't have as direct subgraph k+1-critical graph. Let G be without direct subgraph k+1-critical graph, $\chi(G) \not< k$. Then eliminating edges we must stop to subgraph that is k-critical but not k+1-critical.

Theorem 20. *Let* $n < k(k+1)/2$. *Graph is n-free-k-chromatic iff it does not contain as direct subgraph k+1-critical subgraph minus* n *edges and* $\chi(G) \not< k$.

Proof. Let G has property nfk. Then, if assuming opposite it had k+1-critical subgraph minus n edges, adding n edges gave graph that where not k-chromatic. Trivial.

Let us assume that graph G doesn't contain such subgraph that were k+1-critical minus n edges, but $\chi(G) \not< k$. We must prove that graph has property nfk. Let us assume opposite. Graph doesn't possess property nfk and adding n edges gave k+1-chromatic

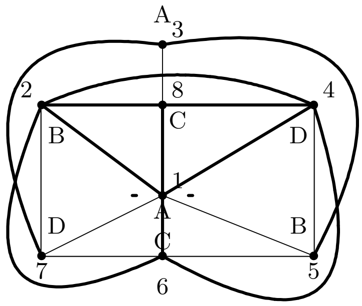

Fig. 8. Example of graph with unique coloring, graph B_7 from projective plane forbidden minors. Bold is drawn edges that go into subgraphs K_5^-. It is easy to sea that graph may be rebuilt using algorithm given in [10] and[4], page 12: K_4 {1,4,5,6} and sequence of vertices 2, 8, 7, 3 do this.

graph [where number of deleted edges satisfies $n < k(k+1)/2$]. In that case we could start taking off edges, not moving n edges that were added, until graph is k+1-critical. It is possible, because in case all possible edges are removed remaining n edges can't give subgraph K_{k+1} because of condition on number n, $n < k(k+1)/2$, that does not allow K_{k+1} be created. Removing added n edges we receive subgraph of G that is k+1-critical minus n edges, and come to contradiction.

4.1 Unique coloring graphs and subgraphs

To go forward we need to consider graphs with forced or unique coloring. Uniquely colorable graph is such that has only one partitioning of colored vertices into sets of one color. See [4] on this matter. Fig. 8 shows an excellent example of such graph, graph B_7 from forbidden subgraphs of projective plane. This graph has as subgraph K_5^- three direct subgraphs and thus it is forced to have only one coloring.

There is simple algorithm that reconstructs every uniquely 4-colorable graph. Take a subgraph K_4 and add successively new vertex of degree three. This fact follows from the fact that every edge 3-colorable graph with unique coloring must have at least one triangle. This fact is know as Fiorini-Wilson-Fisk conjecture and is proved by Robin Thomas and Thomas G. Fowler in [4]. We would refer to it as FWF conjecture. In the example of fig. 8 as K_4 generating set should be taken $\{1, 4, 5, 6\}$ and as all graph generating sequence sequence of vertices $2, 8, 7, 3$.

FWF conjecture deals with planar graphs, but our example B_7 is non-planar. This example shows that conjecture might work outside planarity too. We are not going to try to find out how far it reaches, but use this fact as follows: let us call graph FWF-reconstructible if it may be reconstructed by FWF algorithm. Thus, we know, according [4], that FWF-reconstructibles are planar graphs and some non planar too, for example, B_7. Trivial FWF-reconstructible graph let us call graph K_4.

Lemma 21. *If 4-chromatic graph is nontrivial FWF-reconstructible, then it contains subgraph K_5^-.*

Proof. Let start FWF construction, take K_4 and add vertex. This graph is K_5^-.

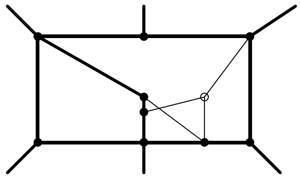

Fig. 9. Forbidden minor for projective plane E_{27} [5]. Graph has at least three cycles of order five, but not triangles. The part that causes odd cycles is drawn bold. Non bold edges does not affect odd cycles in the graph. This graph is 3-chromatic, but free-3-chromatic too. Bold subgraph is of course free-3-chromatic. Added edges can be colored arbitrary, thus, graph retains its free-colorability's feature.

Theorem 22. *If graph is nontrivial FWF-reconstructible, then each its vertex goes into some minor K_5^-.*

Proof. In case graph is planar, the FWF-construction gives only triangulations, and every new vertex fall into some subgraph, not only minor, K_5^-. In case of non-planarity first built K_4 plus last vertex with three edges are into minor K_5^-, because three distinct paths should be from last vertex to one vertex of K_4, because graph is always 3-connected.

We should have use of some reverse assumptions.

Theorem 23. *If 4-chromatic planar graph does not have subgraph K_5^-, then it is not uniquely colorable. More over, no part of the graph taken as induced subgraph may be uniquely colorable graph.*

Proof. First part of assertion follows from fact that uniquely 4-colorable planar graph is FWF-reconstructible. Second part follows straight in case subgraph is 4-chromatic. In case it most 3-chromatic, one excessive color may be interchanged with whatever color from these most three colors.

4.2 Direct subgraphs

It is easy to see that graphs we have constructed as 4-critical graphs does not contain K_5^-, though have it as minor, thus, they may not be uniquely colorable graphs and no part of graph taken as some subgraph may be colored uniquely. We want to conclude from there that k-critical graphs all have property not containing as subgraphs k+1-critical graphs minus edge.

Further we expect that planar 4-critical graph has as direct subgraph $K_4^{\odot}$ which is K_4 with split off edge. General proposition is in Conjecture 35

Let take for example graph E_{27} from forbidden graphs of projective plane. It is 3-chromatic and is free-3-chromatic, because does not have subgraph K_4^-, but it has of course homeomorphic subgraphs K_4^-, not being atetrahedral graph. The same applies for projective plane forbidden graphs E_{11}, E_6 and F_4 too. Neither of them have K_4^- as direct subgraph and they are 3-chromatic. Thus, they are free-3-chromatic.

Two technical lemmas are needed.

Lemma 24. *Every non-edge in critical graph may be colored with different colors.*

Proof. Let one vertex of this non-edge be colored with the lonely color.

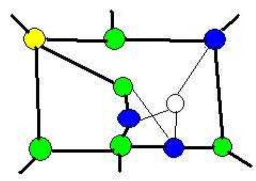

Fig. 10. E_{27} see [5].

Lemma 25. *Every non-edge in critical graph may be colored with the same color.*

Proof. Let one vertex of this non-edge $[vw]$, say, vertex v, be colored with the lonely color. Let recolor other vertex with the same color.

Both lemmas may be generalized.

Lemma 26. *Every independent set of vertices in critical graph may be colored with the same color.*

Proof. Let one vertex of this non-edge $[vw]$, say, vertex v, be colored with the lonely color. Let recolor other vertices with the same color.

Lemma 27. *Every independent set of at most $k-1$ vertices in k-critical graph may be colored with different colors.*

Proof. In case this set of vertices is separating graph into parts, proof follows straightforward. Otherwise let us use induction. Let us split graph along these vertices letting graph be hold together with what it is hold together. As a basis of induction we may take graph with, say, $k+1$ vertices which should be different from K_{k+1}^{-}, which must be absent at all in critical graph.

Lemma 28. *Let in critical graph k distinct vertices for all colorings have k distinct colors, i.e., they color differently, then subgraph K_k is in the graph.*

Proof. Let us suppose opposite and these k vertices have some non-edge. But, according lemma 25, every non-edge in critical graph may be colored with one color, thus, K_k on these vertices must have less than k colors in this coloring.

Lemma before may be generalized. By the way, we must observe that K_k is uniquely colorable graph.

Theorem 29. *In k-critical graph n distinct vertices for all colorings receive n different colors iff induced subgraph on these vertices is K_n.*

Proof. In one direction assertion is trivial. Let us assume that in k-critical graph exist such subgraph induced by n vertices that in all colorings it receive n colors. Let us assume, to the contrary to what theorem says, that this graph is different from K_n and thus has same non-edge vw. But, according lemma there exist coloring were v and w receive different colors, thus, this subgraph is not uniquely colorable graph. Even more, there does not exist pair of vertices that for all colorings would reside in common color subset. Thus, this graph can't even have some subgraph in it that were uniquely colorable, or it is itself uniquely colorable graph K_n. If subgraph has at least one edge, then it must coincide with K_n, starting with K_2. In case it does not have edges, then lemma says that graph can be colored either with one color or at least two colors.

We may generalize the theorem as follows.

Theorem 30. *In k-critical graph G vertices from proper vertex subset S with m vertices for all colorings receive n different colors iff $m = n$ and induced subgraph on these vertices is K_n, or graph G as subgraph has bipartite graph with vertices S in one of its sides.*

Proof. Let arbitrary vertices x and y correspondingly inside and outside of induced subgraph $G(S)$. Then either every such pair must be united by edge to exclude possibility to color, say, x with lonely color and other end recolor with the same color, or every vertex of $G(S)$ by colorings must go into its own color subset, i.e., $G(S)$ is isomorphic to K_n.

Thus, theorem allows case when G is taken odd wheel W_{2n+1} and as S is taken its rim.

Technical lemmas 26 and 27 say that there is possible colorings with any independent set colored either in one color or as many as number of vertices in this independent set not exceeding $k - 1$. If independent set in subgraph is not less than chromatic number of subgraph extracted, i.e. not greater than two, then complementary graph of this subgraph should be tree or forest. Two cases, either subgraph contains odd cycles, $n = 3$, or doesn't, $n = 2$. Technical lemma exclude case when subgraph is independent set. Cases solve trivially.

Corollary 31. *Belonging to its own Hadwiger class k-critical graph has K_{k-1} as subgraph.*

Let us try to prove it for $K < 5$. For case $k = 3$ it is right of course.

Corollary 32. *Planar 4-critical graph has K_3 as its direct subgraph.*

Proof. Let us assume that planar graph exist that do not have K_3. We can't make such graph using Hajós construction, because we would need for this reason minimal block with such property, but just that what is missing. Only critical graphs without this property ar non-planar. Either the construction used in this paper in chapter cannot give graphs without K_3. The graph we are trying to find must be at least 4-connected. ....Proof must proceed.

This fact is known as Grötzsch's theorem which says: every triangle-free planar graph is 3-colorable. See [3] p. 406.

Conjecture 33. *In k-critical graph every vertex either goes in some K_{k-1} or belongs to nontrivial, i.e., with at least three members, independent set of vertices.*

The two famous non-planar 4-critical graphs, Chvátal's graph with 12 vertices and Grötzsch graph with 11 vertices don't have triangles at all, but in both graphs every vertex belongs to independent set with at least three vertices, (see [3] pages 362 and 366).

Corollary 34. *nfk graph does not have as subgraph k+1-critical graph without n edges.*

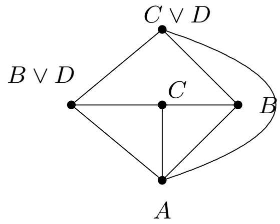

Fig. 11. Wheel graph of order four can't be subgraph of any 4-critical graph.

Conjecture 35. *Belonging to its own Hadwiger class k-critical graph does not have* K_k *as subgraph, but has* K_k^- *or* $K_k^{\odot}$ *as subgraph.*

Proof. Let us give arguments in favor of this supposition. If we do vertices merging operation that subgraph $K_k^{\odot}$ becomes K_k then graph remains k-chromatic but it is not k-critical because vertex split does not affect its chromatic number. Thus, K_k can't be direct subgraph of k-chromatic graph. Doing as if all possible such splits, graph is reduced in some minimal state. Formal proof should establish this, whether it is correct consideration.

If this conjecture where true, then we were to account for derivations of K_{k+1} as possible forbidden subgraphs for k-critical graphs that belong to its own Hadwiger class. Other k+1-critical subgraphs were 'eaten up' by the simplest critical graph, all other critical graph contained as parts this simplest graph, thus not necessary to account in the list of eventual forbidden graphs.

In the construction above where we were building 4-critical graphs, it is easy to see that they are actually 2-free-4-chromatic. I.e., to make subgraph K_5^- in them, two edges at least should be added.

We show it there.

Lemma 36. *4-critical graph does not have* W_4 *as subgraph.*

Proof. . W_4 is uniquely 3-colorable graph. Hub of the wheel may be removed without affecting other coloring of graph in all colorings, except these where probably hub vertex were colored using lonely color. Let us in these cases in place of all vertex with edges as spokes of the wheel remove only one spoke which can't affect these colorings.

Lemma gives what follows.

Theorem 37. *Every 4-critical graph is at least 2-free-4-chromatic.*

Proof. Lemma before shows that 4-critical graph can't have W_4 as subgraph. Neither other case, W_3 plus C_3, that is other possible division of K_5^- minus edge, is possible in 4-critical graph. These two cases exhaust ways of K_5^- minus edge division into components. Thus, 4-critical edge does not have K_5 with two eliminated edges, and it is 2-free-4-chromatic according theorem 34.

But some classes of 4-critical graphs may be 3-free-4-critical too. Not-3-connected 4-critical graphs built above are such. Graph from class of three-connected graphs built above characterized by $3 - 3 - 3$ is 3-free-4-critical, see fig. 13.

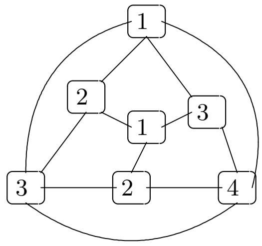

Fig. 12. 4-critical graph characterized by $3 - 3 - 3$ is 3-free-4-critical, because it does not contain any subgraph K_5 minus three edges.

We know that 4-critical graphs can't contain K_k^-, even stronger condition being working. We are interested in question about other k+1-critical graphs minus minimum n edges possibly being present in k-critical graphs. What we know is that for K_5 $n = 3$. What is the general number $n(4)$?

We must the same question put on 5-critical graphs against 6-critical graph minus possible minimum n edges presence in them. What is the number $n(5)$, and can't knowledge of these facts lead to ultimate proof of 4CT?

5 On supposed counterexample to 4CT

Let for $\chi(G)$-critical graphs G function $n : N \to N : \chi(G) \mapsto n$, where n is minimal that G can't contain any $\chi(G) + 1$-critical graph minus n edges.

Ordinary planar graphs that are not 4-critical all are free-4-chromatic with respect set of planar edges but not with respect all, i.e., non-planar edges too. In best case it may be free-4-chromatic but not free^2-4-chromatic. What does it mean? Critical graphs may be built from free-planar parts with some amount of closures. If in place of free-planar graphs graphs from class PP are taken, it can not affect result for 4-critical graphs, because all non-planar 3-blocks coming as ballast should be eliminated if we want obtain 4-critical graph. If we wanted to use PP graphs essentially we would get only non-planar graphs. Is there some class between PP and free-planar graph class that could spoil situation around four color theorem? Free-Hadwiger five class. But it comes only with some special cases with comparing with class PP. Thus, free-planar graph class is that that guides ways of building of class of 4-critical graphs. But this was under assumption that four color theorem is right. We need other way considered too.

If four color theorem is supposedly wrong then smallest counterexample 5-critical planar graph H must be free-5-chromatic at least. May it be built from free-planar graphs and then closures applied?

Let us find planar subgraphs of K_6 with minimum edges eliminated.

Lemma 38. *H must be at least free^3-5-chromatic.*

Proof. Smallest planar subgraphs of K_6 are obtainable eliminating at least three edges.

But two more edges may be more than possible in K_6 subgraphs in H, because just two edges are excessive for graph to be obtainable from free-planar graphs.

It looks like following proposition must be true. We are going to check it further.

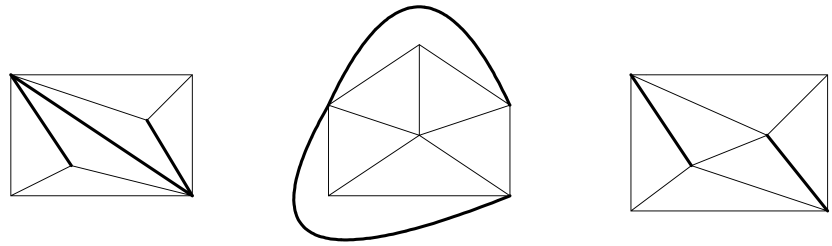

Fig. 13. Cases of subgraphs of K_6 to be considered. Subcases with bold edges not present are to be considered too.

Lemma 39. *H must be at least free^5-5-chromatic.*

Proof. Smallest free-planar subgraphs of K_6 are obtainable eliminating at least five edges.

Next, cases shown in fig. 13 must be considered to the effect that they are not possible in 5-critical graph.

We must end with theorem, *in spe.*

Theorem 40. *Planar 5-critical graph H does not exist.*

Proof. Proof should be based on excluding cases from above.

6 Acknowledgements

I would like to thank professor Jan Kratochvíl for giving me possibility to work in the Department of Applied Mathematics of Charles University in Prague in spring of 2008.

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