Kuratowski Theorem from below.

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Abstract

This note proves the Kuratowski theorem from below, i.e. assuming that Kuratowski-like theorem for Free-Planar graphs is right.

Graph is defined as a pair of sets (V, E), where V is the set of vertices and E – the set of edges. For graph G V(G) is its vertex set and $E(G)$ is its edge set. We denote by $G - e$ graph obtained by deleting edge $e \in E(G)$ from G. Similarly, $G - v$ is graph obtained by deleting vertex $v \in V(G)$ from G.

Similarly, $G.e$ is graph obtained by contracting edge $e \in E(G)$ in G. Reverse operation to edge adding and its contraction is *vertex split* operation $G \odot u$, that is not unique.Thus, if in G by adding and contracting $e \notin E(G)$ appears a new vertex $u \in V(G')$ then there exists such vertex split $G' \odot u$ that we get back previous graph G.

H is *subgraph* of G (denoting it $H \subset G$) if there is such a graph H' isomorphic to H and $V(H') \subset V(G)$ and $E(H') \subset E(G)$.

H is a *minor* of G (denoting it $H \prec G$) if H can be obtained by edge contractions from some subgraph of G. It is easy to see that if $H \prec G$ then H can be obtained from G by vertex deletions, edge deletions and edge contractions.

A class of graphs A is called *minor closed* if for each graph H belonging to A and arbitrary graph G from $G \prec H$ follows that G is in A.

For a minor closed class A, $F(A)$ is the minimal set of forbidden minors, i.e.

$$F(A) = \lfloor \{G \mid G \notin A\} \rfloor.$$

Here we use a notion $\lfloor B \rfloor$ denoting set which contains only minimal minors of B:

$$\lfloor B \rfloor \triangleq \{G \mid H \in B \wedge H \prec G \Rightarrow H \cong G\}.$$

Analogously, $\lceil B \rceil$ which contains only maximal graphs, i.e. all graphs of B that are minors of $\lceil B \rceil$ is defined as follows:

$$\lceil B \rceil \triangleq \{G \mid H \in B \wedge G \prec H \Rightarrow H \cong G\}.$$

Proposition 1. *For a minor closed class A if G doesn't belong to A there exists such $H \in F(A)$ that $H \prec G$ and conversely.*

Theorem 2. *(Robertson, Seymour) $F(A)$ is finite for any minor closed A.*

Let $\mathcal{G}(n, m)$ denote the set of all graphs with $n, (n > 0)$ vertices and $m, (m \geq 0)$ edges. Let $\mathcal{B}(n, m)$ denote the set of all bigraphs with $n, (n > 0)$ vertices and $m, (m \geq 0)$ edges. Let $\mathcal{C}(n, m)$ be arbitrary subset of $\mathcal{G}(n, m)$ and $\mathcal{D}(n, m)$ be arbitrary subset of $\mathcal{B}(n, m)$. Sometimes we are saying that graph G with n vertices and m edges belongs to a set of graphs (not specifying the set) meaning by this set $\mathcal{G}(n, m)$.

Sets of graphs $\mathcal{C}_1 = \mathcal{C}(n_1, m_1)$ and $\mathcal{C}_2 = \mathcal{C}(n_2, m_2)$ are *non-compatible* if no graph from $\mathcal{C}_1$ is a minor for any graph from $\mathcal{C}_2$ and vice versa.

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Lemma 3. *Sets of graphs $\mathcal{G}_1 = \mathcal{G}(n_1, m_1)$ and $\mathcal{G}_2 = \mathcal{G}(n_2, m_2)$ are non-compatible iff $n_1 > n_2$ and $m_1 < m_2$ or vice versa: $n_1 < n_2$ and $m_1 > m_2$.*

Proof. If there holds $n_1 > n_2$ and $m_1 < m_2$, then it is easy to see that compatibility is impossible.

Let us assume that $\mathcal{G}_1 = \mathcal{G}(n_1, m_1)$ and $\mathcal{G}_2 = \mathcal{G}(n_2, m_2)$ are non-compatible. If graphs with equal number of vertices were allowed then we could take a graph and its subgraph with the same number of vertices and they were compatible. If graphs with equal number of edges were allowed then we could take two equal graphs and one replenish with as many as necessary isolated vertices and they were compatible. Thus, the non-compatibility condition must be just as stated by lemma. □

It is easy to see that the statement of the lemma is right also for pairs of sets $\mathcal{B}(n_1, m_1)$ and $\mathcal{B}(n_2, m_2)$ and $\mathcal{B}(n_1, m_1)$ and $\mathcal{G}(n_2, m_2)$ respectively. Evidently, the last case corresponds to Kuratowski graphs, i.e. $\mathcal{B}(6, 9)$ and $\mathcal{G}(5, 10)$ are non-compatible sets of graphs [or actually pair of graphs].

For sets of graphs $\mathcal{G}$ and $\mathcal{S}$ let $\mathcal{G} \mid_{\mathcal{S}}$ denote only these graphs from $\mathcal{G}$ that do not have graphs from $\mathcal{S}$ as minors:

$$\mathcal{G} \mid_{\mathcal{S}} \triangleq \{G \mid G \in \mathcal{G} \wedge \forall H \in \mathcal{S} : H \not\prec G\}.$$

$N_\circ(B)$ denotes the minor closed class with B as its set of forbidden minors, i.e.

$$N_\circ(B) \triangleq \{G \mid \forall H \in B : H \not\prec G\}.$$

In other words, we may say, that $N_\circ(B)$ is a minor closed class generated by its forbidden minors in B. For example, $N_\circ(K_5, K_{3,3})$ is the class of planar graphs, as it is stated by Kuratowski theorem.

Let us define the set $T_\circ(B; C)$ as maximal set with B as its upper bound but C as the set of minors that are forbidden:

$$T_\circ(B; C) \triangleq \{H \mid \exists H_1 \in C \wedge \exists H_2 \in B : H_1 \prec H \prec H_2\}.$$

One more denotation for the minor-closure of B:

$$\langle B \rangle \triangleq \{G \mid \exists H \in B : G \prec H\}.$$

If $B = \{G\}$ then we write $\langle G \rangle$ in place of $\langle \{G\} \rangle$.

Let us use some denotations for some small graphs:

Z_k, $k > 0$, is empty graph with k vertices. For example, $Z_1 = K_1$ and $Z_k = k \times K_1$ for $k > 0$.

$X = K_{1,4} + K_1$, i.e. star with four edges and an isolated vertex.

$Y = K_{1,3} + K_{1,1}$, i.e. star with three edges and an isolated edge.

$V = K_3 + 3 \times K_1$, i.e. triangle and three isolated vertices.

U is cycle of length 5 plus a vertex of degree 2 connected to two non-consecutive vertices of the five-cycle.

Let us state some simple facts with these and Kuratowski graphs:

Lemma 4. $\lceil N_\circ(\{V, Z_7\}) \rceil = \{K_{3,3}, K_5, K_{2,4}, K_{1,5}, U\}$.

Proof. If graph has 6 vertices it is without triangles, i.e. it is a bigraph. There are four possible maximal such graphs, i.e. $K_{3,3}, K_{2,4}, K_{1,5}$ and U.

If graph has less than 6 vertices, it is arbitrary otherwise, i.e. maximal graph is K_5. □

Lemma 5. $\lceil N_\circ(\{V, X, Y, Z_7\}) \rceil = \{K_{3,3}, K_5\}$.

Proof. X excludes $K_{1,5}$ and Y excludes U and both X and Y exclude $K_{2,4}$. □

If Z_7 is removed from lemmas condition then graphs with components isomorphic to Kuratowski graphs and their subgraphs are allowed.

A planar graph is called free-planar, if after adding an arbitrary edge it remains to be planar. In [4] it is proved, that the class of free-planar graphs is equal to $N_\circ(K_5^-, K_{3,3}^-)$, and its characterization in terms of the permitted 3-connected components is given.

In [2] a generalization of the notion of free-planar graphs is suggested. We denote by $Free(A)$ the class of graphs that consists of all graphs which should belong to A after adding an arbitrary edge to them. It is easy to see, that, if A is minor closed, then $Free(A)$ is minor closed too [2]. Because of this we use to say, that $Free(A)$ is *free-minor-closed-class* for a minor closed class A.

In [2] Kratochvíl proved a theorem:

$$F(Free(A)) = \lfloor F(A)^- \cup F(A)^\odot \rfloor,$$

where

$$B^- \triangleq \{G - e \mid G \in B, e \in E(G)\}$$

and

$$B^\odot \triangleq \{H \mid H \cong G \odot v, G \in B, v \in V(G)\}.$$

We are going to apply $Free$ to a graph as sort of operator until only isolated vertices are left there. We denote by $Free^k(A)$ repeatedly applied $Free$ k times, i.e.

$Free^0(A) = A$;

$Free^k(A) = Free(Free^{k-1}(A))$.

Let for a minor closed class A $Free^m(A)$ is not consisting of only empty graphs but $Free^{m+1}(A)$ is, then we say that A is *of depth m*.

In the graph G with some vertex v a vertex split $G \odot v$ is called *proper* if both new vertices arising in the result of the split of v are of degree at least two. Otherwise the vertex split is called *non-proper*.

Theorem 6. *Let a class A be of depth m and all graphs of $F(A)$ belong either to mutually non-compatible or coinciding sets of graphs. If there holds*

$$F(Free(A)) = F(A)^-$$

then there holds also

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$.

Proof. By induction if m is equal to 1 all is done by theorem's assumption otherwise it suffices to prove that

$$F(Free^{k+1}(A)) = F(Free^k(A))^-$$

for $1 \le k < m$ assuming that

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

is right.

Let us suppose first that $F(Free^k(A))$ consists only from one graph. Then all graphs $F(Free^k(A))^-$ have the same number of edges and they can't be proper minors of each other. Thus, they all are present in $F(Free^{k+1}(A))$.

$F(Free^k(A))^\odot$ can not give some contribution to $F(Free^{k+1}(A))$ either. Let us suppose for a moment that it does and some graph $G \in F(Free^k(A))$ is such that G' with some vertex v split ($G' = G \odot v$ giving new vertices v_1 and v_2) were not present in $F(Free^k(A))^-$.

Let us suppose that this vertex split was non-proper. Then a corresponding hanging edge or isolated edge arises, but graph without this edge is already present in $F(Free^k(A))^-$. Thus, non-proper vertex split can not give any new contributor to $F(Free^{k+1}(A))$. Further, let us suppose that this vertex split was proper. Let us find some ancestor H of G in $F(Free^{k-1}(A)$ such that H minus some edge e is equal to G. Let us add $e = (s, t)$ to G' getting a new graph $H'(= G' + e')$ in this way: if none of ends of e were equal to v then adding $e(= e')$ is possible only in one way; if one of the ends, say s, of e is equal to v then without loss of generality we add a new edge $e' = (v_1, t)$ to one of the new vertices (i.e. v_1) arising in the result of the split of v. Then this graph must be also contributor to $F(Free^k(A))$, because there exists some vertex split $H \odot v$ such that $H \odot v$ is equal to H' but the assumption of the theorem excludes this [...because H' has a proper minor h already present in $F(Free^k(A))^-$. If $e' \in E(h)$ then $h - e'$ should be also a minor of G', thus G' is not a contributor in $F(Free^{k+1}(A))$. If $e' \notin E(h)$ then h minus arbitrary edge is minor of G' and present in $F(Free^k(A))^-$. Thus, in this case too G' is not a contributor in $F(Free^{k+1}(A))$].

Let us suppose that$F(Free^{k-1}(A)$ consists from many graphs. But, because of either non-compatibility or coincidence of the sets to which these graphs belong the same is true for the set $F(Free^k(A))$ too. Truly, non-compatible assumed contributors of $F(Free^k(A))^-$ can not exclude each other. Either can not such that are with equal number of vertices and edges except the cases of isomorphism. Thus distinct descendants of the same level from distinct forbidden graphs can not be proper minors of each other.

The consideration about the contribution of $F(Free^k(A))^{\odot}$ does not change in the general case. □

Theorem 7. *For $A = Planar$ and $0 < k \leq 10$*

$$F(Free^k(Planar)) = \mathcal{B}(6, 9-k) \mid_{\{X,Y\}} \cup \mathcal{G}(5, 10-k).$$

Proof. Graphs X, Y exclude these minors that are present in $K_{2,4}$ but are absent in $K_{3,3}$. □

Theorem 8.

$$\langle \{K_{3,3}, K_5\} \rangle - \bigcup_{k=0}^{10} F(Free^k(Planar)) = T_{\circ}(\{K_{2,4}\}, \{X, Y\}).$$

Proof. The same what in previous theorem in other terms. □

Corollary 9. *For $A = Planar$ Kratochvíl's theorem has following appearance*

$$F(Free^k(Planar)) = F(Free^{k-1}(Planar))^-$$

for $k = 1, ..., 10$.

Let for a class A of depth m holds:

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$. Then we call this class $\mathcal{M}$-class.

Then we can state:

Corollary 10. *For the case A is M-class of depth m Kratochvíl's theorem has following appearance*

$$F(Free^k(A)) = F(Free^{k-1}(A))^-$$

for $k = 1, ..., m$.

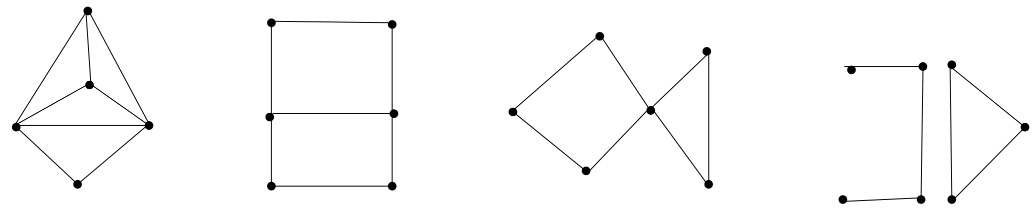

Figure 1: Non-trivial example of $\mathcal{M}$-class: Forbidden graphs of an $\mathcal{M}$-class A with one forbidden graph from $\mathcal{G}(5,8)$, two forbidden graphs from $\mathcal{G}(6,7)$ and one forbidden graph from $\mathcal{G}(7,6)$.

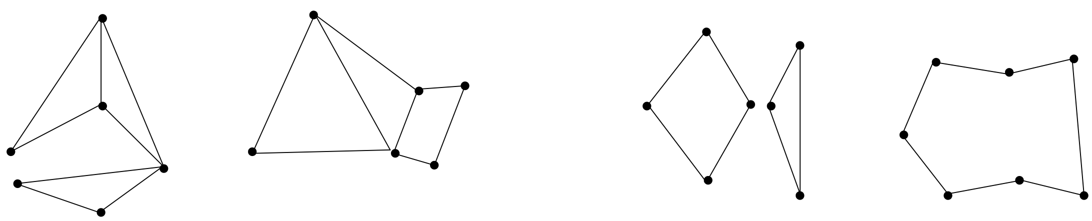

Figure 2: Non-trivial example of $\mathcal{M}$-class: Instances of the forbidden graphs of A from fig. 1 after proper vertex split operation is applied to them.

Theorem 11. *Let $B = free(A)$ and B is $\mathcal{M}$-class. A is $\mathcal{M}$-class iff $F(B) = F(A)^-$.*

Proof. If A is $\mathcal{M}$-class then theorem by definiton of $\mathcal{M}$-class $F(B) = F(A)^-$. Conversely, from the definition of the $\mathcal{M}$-class and from facts that $F(B) = F(A)^-$ and B is $\mathcal{M}$-class follows that A is $\mathcal{M}$-class too. □

Last theorem applied to theorem 14 from [4] serves as the proof of Kuratowski theorem from below. Let us state it as the following theorem.

Theorem 12. *The class Planar is $\mathcal{M}$-class and equal to $N_\circ(\{K_{3,3}, K_5\})$.*

Proof. From theorem 14 [4] $free(Planar)$ is equal to $N_\circ(\{K_{3,3}^-, K_5^-\})$. In corollary 10 ... together with the definition of $\mathcal{M}$-class we conclude that $Free(Planar)$ is $\mathcal{M}$-class. Thereafter, according theorem 11 class $N_\circ(\{K_{3,3}, K_5\})$ is $\mathcal{M}$-class and equal to $Planar$, [...because $F(Free(Planar)) = F(Planar)^-$??? causing Planar is $\mathcal{M}$-class???].

□

In fig. 1 the forbidden graphs of a non-trivial example of $\mathcal{M}$-class are given. Fig. 2 shows instances of these forbidden graphs after proper vertex split operation is applied to them. Fig. 3 shows the instances of the forbidden graphs of corresponding free class of this $\mathcal{M}$-class where proper vertex split operation is possible. Fig. 4 shows instances of the graphs of Fig. 3 where proper vertex split operation is applied to them.

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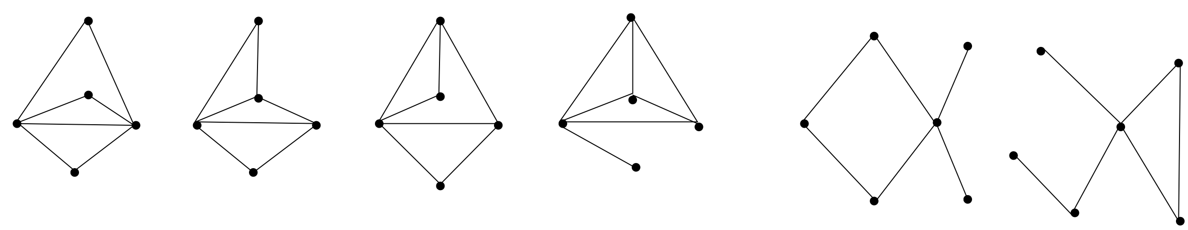

Figure 3: Non-trivial example of $\mathcal{M}$-class: Instances of the forbidden graphs of $Free(A)$ where proper vertex split operation is possible.

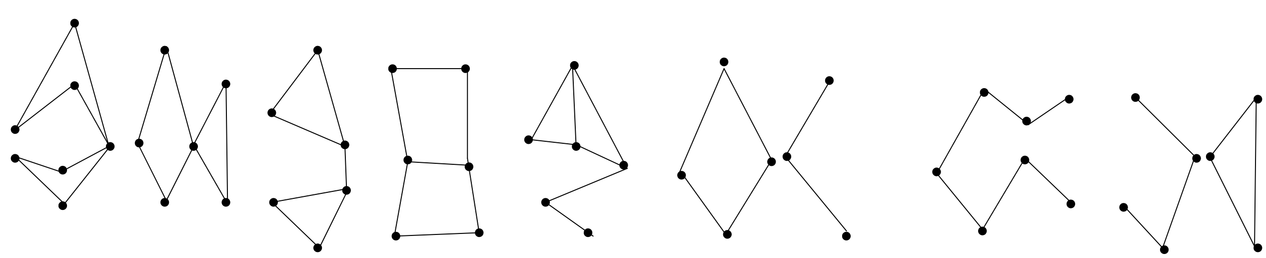

Figure 4: Non-trivial example of $\mathcal{M}$-class: Instances of the graphs from fig. 3 with proper vertex split operation applied to them.

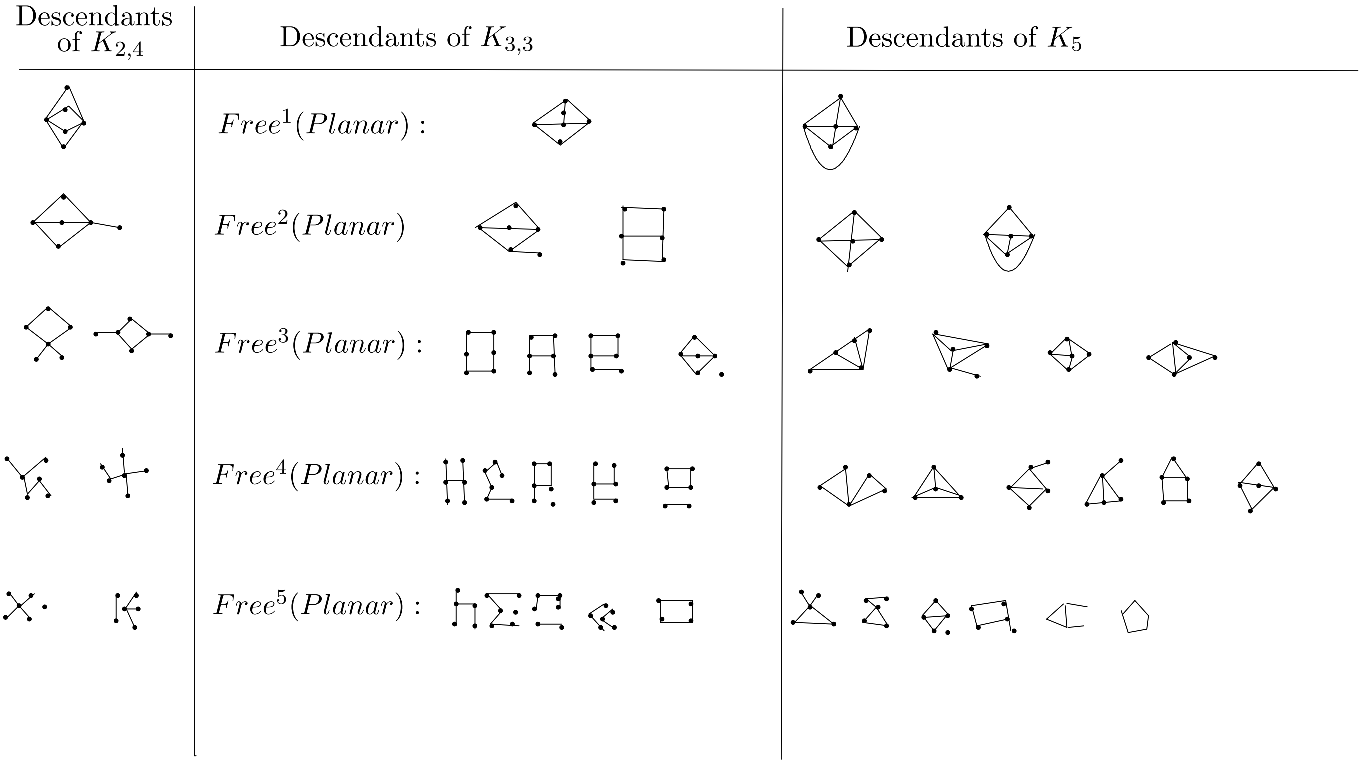

Figure 5: In the second and third columns there are $Free^k(Planar)$, $1 \leq k \leq 5$.

References

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