Kuratowski Theorem from below

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A planar graph is called *free-planar*, if after adding an arbitrary edge it remains to be planar [1]. Here is shown that it is possible to give a proof of a Kuratowski like theorem for the free-planar graphs that almost without additions fits for the planar graphs too.

Theorem 1. *The forbidden minors for the class of free planar graphs are* $K_5^-, K_{3,3}^-$. *The forbidden minors for the class of planar graphs are* $K_5, K_{3,3}$.

Proof. Let us assume that G is planar but not free planar. Then there exists an edge xy not belonging to the graph whom adding to the graph it becomes non-planar. Then in G for an arbitrary cycle C through x, y there exists *a pair of screening bridges* B_x and B_y *x from y with respect to* C, i. e. either B_x and B_y are not placeable on one side against C or they are connected [i.e. not placeable together] with an alternating [i.e. on one and other side of C] sequence $[B_1, ..., B_{2k}, k > 0]$ of non-screening [x from y] bridges.

Let us describe the bridge with the sextet $[x, a, b, y, c, d]$, where values of it are either vertices on the cycle C or logical values $T(= true)$ or $F(= false)$ [see fig. 1]:

1) in the place of $x(y)$ stands T if $x(y)$ is a *leg* [i.e. the touch vertex to C] of the bridge with respect to C, otherwise F;

2) $a(c)$ is the nearest next leg moving clockwise from $x(y)$ before $y(x)$ if any, otherwise F;

3) $b(d)$ is the nearest next leg moving anticlockwise from $y(x)$ before $x(y)$ if any, otherwise F;

The screening condition of a bridge $[x, a, b, y, c, d]$ x from y on C is – the values a, b, c, d are not F. Non-screening bridges $B_i, [0 < i \leq 2k]$ are of the form $[x, a, b, y, F, F]$ or $[x, F, F, y, c, d]$ in general.

There are three simple $[k = 0]$ cases and one non-simple case $[k > 0]$ to be considered:

1) In the first case, for one of bridges, say, B_x both in x and y stand T. K_5^- arises even when B_y is simple: $[T, a, a, T, c, c]$.

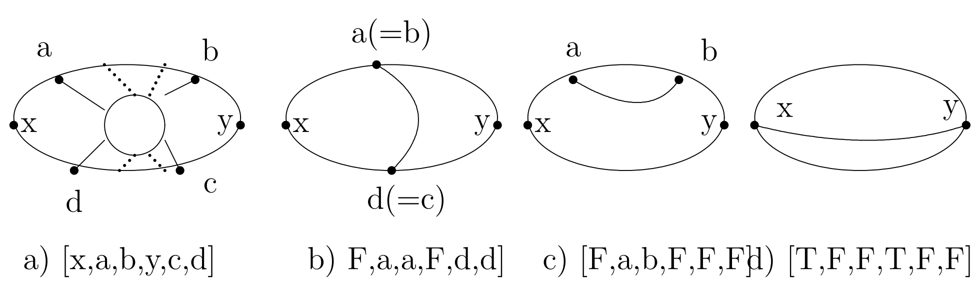

Figure 1: The bridge with respect to the cycle with two distinguished vertices x and y and its characterizing sextet: a) the bridge in general; b)a simple screening bridge ; c) a simple non-screening bridge with legs distinct from x, y ; d) edge x, y as a simple bridge with respect to C.

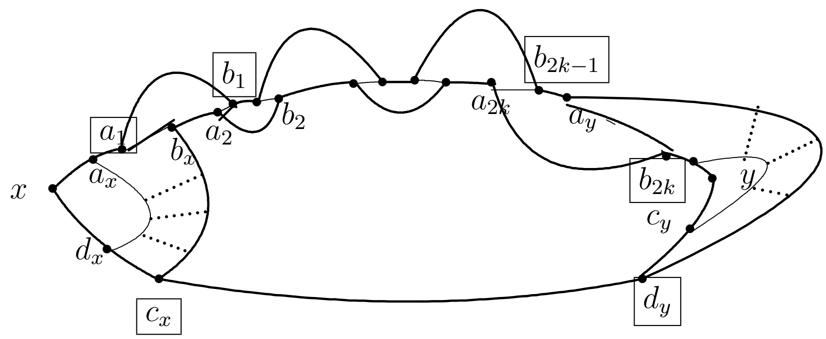

Figure 2: Subgraph with reduced Kuratowski minor K_5^- in correspondence drawn bold.

2) In the second case, in place of, say, x stands T for both B_x and B_y. $K_{3,3}^-$ arises: simplest case, with minimal number of edges – both bridges are of the form $[T, a, a, F, c, c]$, giving $K_{3,3}^-$ with two redundant edges and one subdivided edge.

3) In the third case, B_x characterized by $[x, a_x, b_x, F, c_x, d_x]$ and B_y by $[F, a_y, b_y, y, c_y, d_y]$ are not placeable on one side of C, non satisfying legs' non intersecting condition – existence of, say, two paths $x..a_x.b_x.a_y.b_y..y$ and $x..d_x.c_x.d_y.c_y..y$ through edges of C. Then easy checkable [in case of minimum of edges just] $K_{3,3}^-$ arises.

4) In the case $k > 0$, two bridges B_x and B_y can not be placed on one side of C, if alternating sequence of bridges of form, say, $[F, a_i, b_i, F, F, F]$ $[0 < i \leq 2k]$ join them satisfying the condition – existence of a path $x.a_1..b_x.a_2..b_1.a_3..$ ... $.a_{2k}..b_{2k-1}.a_y..b_{2k}.y$ through edges of C.

When the bridges B_x, B_y joining condition is true, after contracting $c_x..d_y$ a wheel with an extra edge W_4^+ $[K_5^-]$ on vertices $c_x, a_1, b_x, a_{2k}, a_y$ [in case of minimum of edges] arises [see fig. 2]:

1) the spikes of the wheel: $c_x..a_1, c_x..b_x, d_y..a_{2k}$, $d_y..a_y$;

2) the rim of the wheel: $a_1..b_2.a_2..b_2.a_3..$ ... $.a_{2k}..b_{2k}..a_y.b_{2k-1}..a_{2k-1}.$ ... $.b_3..a_3.b_1..a_1$;

3) the extra edge: $a_2..b_1$;

Thus G must have at least one of the reduced Kuratowski graphs as its minor and the proof of the Kuratowski like theorem for free planar graphs is completed. It remains to check that in all cases when reduced Kuratowski graphs as minors arose, together with xy it gave rise to Kuratowski graph as minor too. This completes the proof of the Kuratowski theorem for all class of the planar graphs. □

References

[1] D. Zeps, *Free Minor Closed Classes and the Kuratowski Theorem*, KAM Series, 98-409, Prague, 1998, 10 pp.